Optimal Replacement Policies
for a Multistate System

by

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Abstract

In this paper replacement rules for a deteriorating, multistate system with states $0,1,2,\ldots,M$ are studied. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0.

It is assumed that there is associated a cost b_i when the system is in state i and a cost c_i for a replacement from state i.

The main replacement policies discussed here include the control limit rules, i.e. there exists a $k \in \{0,1,\ldots,M-1\}$ such that the system is replaced as soon as it reaches one of the states $0,1,\ldots,k$.

For each $k \in S \subseteq \{0,1,\ldots,M-1\}$ we consider well known binary replacement policies, such as age replacement policy, block replacement policy and the replacement policy treated by Bergman (1978) where the deteriorating process depends on an underlying state variable representing wear, accumulated damage or accumulated stress, etc.

The expected average long run cost and the total discounted cost are minimized determining optimal rules. Furthermore, some generalizations are given.

Contents

1. Introduction

For nearly two decades, there has been a large and continuing interest in the study of maintenance and replacement of stochastically failing equipment.

The motivation for this research has largely been the growing importance of complex electronic equipment in both industrial and military activity.

Lately, however, new applications have arisen in such areas as health, ecology and environment.

There are many ways to classify the works in maintainability; important factors are (Pierskalla and Voelker (1976)):

i) States of the system, such as deterioration level, age, number of spares, number of state variables, etc.

ii) actions available, such as repair, replacement, opportunistic replacement (two or more repair activities done concurrently may cost less than if they are done separately), continuous monitoring, discrete inspections, destructive inspections, etc.

iii) the time horizon involved, such as finite or infinite and discrete or continuous.

iv) knowledge of the system, such as complete knowledge or partial knowledge, the latter involving for example noisy observation of the states, unknown costs, unknown failure distributions, etc.

v) objectives of the system, such as long run expected average cost per unit time and expected total discounted cost, both of which should be minimized.

We now review some of the better known maintenance/replacement models which we feel are of interest for our purposes in this paper and the situations they describe.

The classification has been done in such a manner that it is easy to see the connection with our models.

A. Maintenance models formulated as Markov-decision models

A system is inspected at discrete points of time and a decision is made to repair or replace the system whenever it is found to be in a certain set of states. In the absence of a decision to repair or replace, it is assumed that the system deteriorates stochastically through a finite set of states denoted by the set of integers $\{0,1,\dots,M\}$ according to a Markov chain.

The state M denotes a new or completely renovated system and the state 0 an inoperative or failed system. After inspection of the system a decision is made to repair, replace or do nothing.

In most of the models there are only two possible decisions at any inspection point, decision 1 means "do nothing" and decision 2 means "replace".

The cost to replace a system that has not failed, c_1, and a higher cost, c_2, to replace a failed system, are the only costs involved.

Upon inspecting the system at any time, it is possible to replace the system before failure. In this way it may be possible to avoid the consequences of failure or further deterioration of the system.

Generally we call a replacement before failure a "preventive replacement".

Derman (1963,b) has shown under some assumptions on the transition probabilities that the optimal replacement rule obtained by minimizing the expected long run average cost per unit time is a "control limit" rule; that is, there is a state $k \in \{0,1,\dots,M-1\}$ such that if the observed state i satisfies $i \leq k$ then replace the system and if $i > k$ do nothing. k is called the "control limit". The same key result holds when the objective is changed to minimize the total long run discounted cost.

The control limit rule is also optimal in the long run average cost case with the cost function generalized to allow an "occupancy" cost b_i, associated with being in each state i $(0 \leq b_M \leq b_{M-1} \leq \cdots \leq b_0)$, Kolesar (1966).

If the replacement problem is modelled as a finite state Semi-Markov process, the control limit rule is optimal in the long run average cost case under some assumptions on the cost functions and the expected sojourn times in each state, Kao (1973).

B. Age dependent replacement models and Block replacement models in two-state systems

Consider a two-state system whose replacement upon failure cost c_1 and whose replacement before failure costs $c_2 < c_1$. The replacement time is negligable.

It has been shown by Barlow and Proschan (1965) that if F, the distribution of time to failure, has a strictly increasing failure rate, then there exists a unique T^* such that the expected cost per unit time is minimized if the system is replaced at age T^* or at failure, whichever occurs first.

Various modifications and extensions of this basic model have been made.

Fox (1966) discussed the age replacement policy when the object--function was the total expected discounted cost of maintenance. Sheaffer (1971) introduced an age dependent cost in the model to reflect the decrease in efficiency of the system with age.

Berg (1976) proves that the age replacement policy itself is the optimal decision rule amongst a wide class of replacement rules.

Barlow and Hunter (1960) introduced a socalled minimal repair policy for a complex system. Here the system is replaced or overhauled at age T at a cost c_1'. Intervening failures are rectified at a cost c_2' through minimal repair which does not alter the failure rate of the system.

The block replacement policy, in which the system is replaced periodically at times nT, $n = 1,2,\ldots,$ and at failures, is commonly used for complex electronic systems such as digital computers, and electical parts such as light bulbs and vacuum tubes.

Barlow, Proschan (1965) and Berg, Epstein (1978) have compared the block replacement policy with age replacement. If there is more than one system (unit) the planned replacement times (nT, $n = 1.2,\ldots,$) are common for all of them. This is why the name block replacement is used.

If c is the cost of a planned replacement of one unit, it is reasonable that c is less than c_2, the replacement cost of a preventive replacement in an age replacement policy. This is at least true when we replace more than one unit at nT, $n = 1,2,\ldots,$.

The main drawback of the block replacement policy (BRP) is that at planned replacement times we might replace practically new items.

The minimal repair policy is a modification of the BRP, other modifications of the BRP have been suggested by Cox (1962), Bhat (1969), Berg and Epstein (1976).

C. Shock models in two-state systems

Here we study replacement policies based on measurements of some increasing underlying state variable representing wear, accumulated damage or accumulated stress etc. The failure time of the system depends on this state variable and upon failure the system must be replaced by a new identical one.

A failure cost is also incurred. If the system is replaced before failure, a smaller cost is incurred.

We replace the system at any stopping time T, based on the underlying process, $\{X(t),\ t \geq 0\}$, before failure time. The process $\{X(t),\ t \geq 0\}$ has been assumed to be a nonhomogeneous Poisson process by A. Hameed and Proschan (1973), a Semi-Markov process by Feldman (1976) and a One sided Lévy process by Zuckerman (1977), etc.

In an important paper by Bergman (1978) the process is a general stochastic process, right-continuous and increasing.[1]

Let ξ be the failure time for the system, then Bergman assumes

$$P\{\xi > t \mid X(s),\ s \geq 0\} = e^{-\int_0^t v(X(s))ds}$$

where $v(\cdot)$ is a right-continuous, non-negative, increasing, finite and real-valued function.

(1) We will in the following use the term "increasing" in place of "non-decreasing" and "decreasing" in stead of "non-increasing".

The optimal replacement rule which minimizes the average long run cost per unit time, is shown to be a control limit rule, i.e. it is optimal to replace at failure or when the state varable $X(\cdot)$ has reached some threshold value, whichever occurs first.

For a system composed of many units, the repair or replacement of one unit should sometimes be considered in conjunction with what happens to the other units. Because of the complexity of these models in general, only a few special situations have been analyzed.

In this paper we will not discuss such policies.

Observe that we have only mentioned a few papers. Several others will be given in the list of references. See also Pierskalla and Voelker (1976) which have 250 references.

The replacement theory up till now has mainly assumed the systems to be binary, however, in many practical situations it is useful to have additional states between the two, good and failed.

Usually the deterioration process is then assumed to be a Markov chain (with stationary transition probabilities) or a Semi-Markov process. We often feel this is an oversimplification of real life since the deterioration after the point of time the system has just entered a state, may be strongly dependent on the age of the system at this epoch.

In this paper we will propose a general multistate model which will take this into account.

2. The model

Consider a system, in use or storage, which is deteriorating and at any instant of time can be in one of a number of possible states $\{0,1,\dots,M\}$. The $M+1$ states represent successive levels of performance ranging from the perfect functioning level M down to the complete failure level 0.

Let $\{Y(t),\ t \geq 0\}$ be a right-continuous and decreasing stochastic process representing the state of the system when no replacement action is performed.

For $k = 0,1,2,\dots,M-1$ introduce the random variables

$$R^k = \inf\{t : Y(t) \leq k\}\ ,\ \theta^k$$

representing respectively the lifelength in the states $\{k+1,\dots,M\}$ and the first state the system reaches among the states $\{0,1,\dots,k\}$.
We will assume $0 < r_k \overset{\text{def}}{=} ER^k < \infty$ for $k = 0,1,\dots,M-1$.
Let $G_k = \{0,1,\dots,k\}$, $G_k^c = \{k+1,\dots,M\}$ and

$$Q_j^k(t) = P\{R^k \leq t,\ \theta^k = j\} \quad \text{for } k = 0,1,\dots,M-1 \text{ and } 0 \leq j \leq k\ .$$

Then $Q_j^k(t)$ represents the probability that the first state in G_k is j and the amount of time in G_k^c is less that t or equal to t.
Obviously

$$F^k(t) \overset{\text{def}}{=} P\{R^k \leq t\} = \sum_{j=0}^{k} Q_j^k(t)$$

$$P_j^k \overset{\text{def}}{=} P\{\theta^k = j\} = Q_j^k(\infty)$$

$$\sum_{j=0}^{k} P_j^k = 1$$

Now if $P_j^k > 0$, let $F_j^k(t) = \dfrac{Q_j^k(t)}{P_j^k} = P\{R^k \leq t \mid \theta^k = j\}$. If $P_j^k = 0$, let $F_j^k(t)$ be arbitrary. Hence $F_j^k(t)$ represents the conditional probability that the system will move from G_k^c to G_k within an amount of time t given that the first state in G_k is j.

P_j^k represent of course the probability that the first state in G_k is j

(We have used a similar notation as Ross (1970) page 85.)

Note that the assumption $ER^k < \infty \Leftrightarrow E\{R^k | \theta^k = j\} < \infty$ for all $j \leq k$ such that $P_j^k > 0$.

Let $h_i^k = \int_0^\infty u\, dQ_i^k(u)$ for $i \leq k$. Then $h_i^k < \infty$ since $h_i^k = P_i^k \int_0^\infty u\, dF_i^k(u) = P_i^k E\{R^k | \theta^k = i\} < \infty$ if $P_i^k > 0$ and $h_i^k = 0$ if $P_i^k = 0$.

Thus the deterioration process is governed by the distributions:

$$Q_j^k(t) \quad \text{for} \quad k = 0,1,\ldots,M-1 \quad \text{and} \quad j \leq k \quad \text{or equivalently by}$$

$$F_i^k(t),\ P_i^k \quad \text{for} \quad k = 0,1,\ldots,M-1 \quad \text{and} \quad i \leq k.$$

If we have absolutely continuous distributions, then we define $q_j^k(t) = \frac{d}{dt}Q_j^k(t)$, $f_j^k(t) = \frac{d}{dt}F_j^k(t)$, $f^k(t) = \frac{d}{dt}F^k(t)$ for $k = 0,1,\ldots,M-1$.

If we let $g_i^k(t)$ be the probability that the system moves to state i $(i \leq k)$ given that the system moves from G_k^c to G_k at t, we see that

$$Q_i^k(t) = \int_0^t g_i^k(s)\, dF^k(s), \quad k = 0,1,\ldots,M-1 \quad \text{and} \quad i \leq k.$$

Hence $q_i^k(t) = g_i^k(t) f^k(t)$.

We will in the following use the notation $\bar{F}^j(t) = 1 - F^j(t)$.

When the system is in state i there is associated a cost b_i per unit time.

If the system is replaced by a new system when the system is in state i, this cost equals c_i. We will assume $b_M \leq b_{M-1} \leq \cdots \leq b_0$, $c_0 \geq c_1 \geq \cdots \geq c_M > 0$ and that all costs are finite.

Observe that these costs may depend on which replacement rule we use.

The costs are average costs in each state. Obviously a replacement cost is positive, even a replacement from the best state M is connected with some loss. Consider for example the replacement of a perfect functioning machine with a new. We assume the b_i's to be positive, but as we will see, this is no restriction.

It is assumed that the costs b_i, c_i are independent of the system age; the states of the system give all information about different costs. In chapter 8 we will show that the model easily can be extended to include age-dependent costs.

Our problem is to find a replacement strategy that give "minimum" costs. More precisely we want to minimize <u>the expected average long run cost per unit time</u>, B^δ when a replacement rule δ is used or <u>the expected total discounted cost</u>, B^δ_α where α is a positive discount factor and a cost c incurred at time t is equivalent to a cost $ce^{-\alpha t}$ at time 0.

We may write

$$B^\delta = \lim_{t\to\infty} \frac{1}{t}\Big\{ \sum_{i=0}^{M} b_i \cdot \text{expected time in state } i \text{ in } [0,t]$$

$$+ \sum_{i=0}^{M} c_i \cdot \text{expected number of replacements from state } i \text{ in } [0,t] \Big\}$$

$$B_\alpha^\delta = \sum_{i=0}^{M} b_i \cdot \text{expected "discounted time" in state } i \text{ in } [0,\infty)^{(2)}$$

$$+ \text{ expected discounted cost due to replacements in } [0,\infty)\ .$$

We now contend that it is no restriction to assume b_i to be positive in order to find the optimal rule δ . To see this, let $b_i^* = b_i + N$ such that $b_i^* \geqq 0 \ \forall\ i$. Then $b_i = b_i^* - N$ and

$$\lim_{t\to\infty} \frac{1}{t}\left\{ \sum_{i=0}^{M} (b_i^*-N) \cdot \text{expected time in state } i \text{ in } [0,t]\right\}$$

$$= \lim_{t\to\infty} \frac{1}{t}\left\{ \sum_{i=0}^{M} b_i^* \cdot \text{expected time in state } i \text{ in } [0,t]\right\} - N$$

since $\sum_{i=0}^{M}$ expected time in state i in $[0,t] \equiv t$.

Similarly for the discounted case:

$$\sum_{i=0}^{M} (b_i^*-N) \cdot \text{expected discounted time in state } i \text{ in } [0,\infty)$$

$$= \sum_{i=0}^{M} b_i^* \cdot \text{expected discounted time in state } i \text{ in } [0,\infty) - N\int_0^\infty e^{-\alpha t}dt\ .$$

The main replacement models discussed here will include the control limit rules. We will then get the states divided by the control limit into the "good" and the "bad" states. In principle we have a binary situation, however the fact that we may have several "good" and "bad" states makes it more complicated. I most cases there will be M possible control limits $(0,1,\dots,M-1)$, but we assume that the control limits that are possible are included in $S \subseteq G_{M-1}$.

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(2) By the "discounted time" in state i in $[0,\infty)$ we mean

$$\int_0^\infty e^{-\alpha t} I(Y(t)=i)dt \quad \text{where} \quad I(Y(t)=i) = \begin{cases} 1 & \text{if } Y(t)=i \\ 0 & \text{otherwise} \end{cases}$$

For a system where the states are known at any point of time, S may be taken G_{M-1}. Consider a system where the state of the system is known only when we replace the system or when it reaches state 0. Then we may take $S = \{0\}$. If state M is the functioning state and the states $\{0,1,\ldots,M-1\}$ are failure modes, we may have $S = \{M-1\}$.

Unless otherwise specified all replacement times are negligible, we will briefly discuss the situation with non-negligble replacement times in chapter 9.

Queuing problems in connection with the facilities necessary to maintain the system, such as manpower, availability of spares are ignored. We assume that it is always possible to replace the system with a new and statistically identical system or repair to as a "good as new" condition.

We shall consider stationary replacement rules only, i.e. strategies such that for each new system the same replacement rule is used.

In a paper by Bergman (1980a) it is proved that a stationary nonrandomized replacement strategy is optimal (i.e. it minimizes the long run expected cost per unit time) for a large class of replacement problems.

The main replacement policies treated here are "multistate age replacement policy" (MARP), "multistate block replacement policy" (MBRP) and "a general multistate replacement model".

The MARP (MBRP) is of course a multistate generalization of the usual binary age replacement policy (block replacement policy).

In MARP we consider rules of the following form: Replace at $\min(R^k,T)$, where T is a constant and $k \in S$. "$T = \infty$" means a control limit rule with control limit k.

If we let $S = \{M-1\}$, we have an age replacement model with M failure modes. In fact, this is the model discussed by Mine and Nakagawa (1978).

In the MBRP the rules are of the form: "Replace at nT, $n = 1,2,\ldots$, and replace as soon as the system reaches the control limit k $(k \in S)$. "$T = \infty$" means the same as in MARP.

The special case $S = \{M-1\}$ corresponds to a block replacement model with M failure modes.

Note that if $y = \inf\{t : \bar{F}^k(t) = 0\}$, we consider these rules for $T \leq y$ only.

The general multistate replacement model is an extension of the work by Bergman (1978).
Here we consider replacement rules of the form: Replace at $\min(R^k, T)$ where T is a stopping time with respect to $X(\cdot)$, an underlying, observable stochastic process representing wear, accumulated damage or accumulated stress, etc. and $k \in S$.

If we let $\tilde{Q}_j^k(t) = P\{R^k \leq t,\ \theta^k = j \mid X(s),\ s \geq 0\}$, then we may define $\tilde{F}_j^k(\cdot)$, $\tilde{F}^k(\cdot)$, $\tilde{P}_j^k$, $\tilde{q}_j^k(\cdot)$, $\tilde{f}_j^k(\cdot)$, $\tilde{f}^k(\cdot)$ and $\tilde{g}_j^k(\cdot)$ similarly to the definitions on page 8 and 9.

Letting $M = 1$, $k = 0$ and $\tilde{Q}_0^0(t) = \tilde{F}^0(t) = 1 - e^{-\int_0^t v(X(s))ds}$

where $v(\cdot)$ is a right-continuous, non-negative, increasing, finite and real-valued function, we get the model treated by Bergman (1978).

Note that in order to minimize B^δ and B_α^δ it will be sufficient to introduce

$Q_i^k(t)$ $(\tilde{Q}_i^k(t))$ for $k \in S$ $0 \leq i \leq k$ and $F^j(t)$ $(\tilde{F}^j(t))$ for $M-1 \geq j \geq \min\{k : k \in S\}$.

Then for $S = \{M-1\}$ we just introduce $Q_i^{M-1}(t)$ $i \leq M-1$ $(F^{M-1}(t) = \sum_{j=0}^{M-1} Q_j^k(t))$ and correspondingly for $S = \{0\}$ $Q_0^0(t)$, $F^j(t)$ for $j = 0,1,\ldots,M-1$.

Thus for $M=1$ we are in the binary case with $Q_0^0(t) = F^0(t)$ as the failure distribution.

In this paper we will give the optimal replacement rule within the classes of MARP, MBRP and the general replacement policy.

Note that the preventive replacement times in MARP and MBRP are determined before observing the system. Wheather this is an objection to these policies, will obviously depend on the particular system at hand.

We will not discuss this any further here.

We now simplify notation and write failure (k) for the event that the system moves or jumps from G_k^c to G_k.

In order to get nice expressions for B^δ and B_α^δ we furthermore introduce for $M-1 \geq k \geq 0$

$$
\begin{aligned}
(2.1) \qquad C^k(Y(x)) &= \sum_{i=0}^{M} c_i I(Y(x)=i) \\
&= \sum_{i=0}^{k} c_i I(Y(x)=i) + \sum_{i=k+1}^{M} c_i I(Y(x)=i) \\
&= \sum_{i=0}^{k} c_i I(Y(x)=i) - \sum_{i=k}^{M-1} \gamma_i^k I(Y(x)>i) \\
&= C_1^k(Y(x)) + C_2^k(Y(x))
\end{aligned}
$$

where we have introduced

$$\gamma_k^k = -c_{k+1} \quad \text{and} \quad \gamma_i^k = c_i - c_{i+1} \quad \text{for} \quad i = k+1,\ldots,M-1$$

$$\text{i.e.} \quad c_i = -\sum_{j=k}^{i-1} \gamma_j^k \quad \text{for} \quad i = k+1,\ldots,M .$$

(2.2) $$C_1^k(Y(x)) = \sum_{i=0}^{k} c_i I(Y(x)=i)$$

(2.3) $$C_2^k(Y(x)) = \sum_{i=k+1}^{M} c_i I(Y(x)=i) = -\sum_{i=k}^{M-1} \gamma_i^k I(Y(x)>i)$$

$I(\cdot)$ is as before the indicator function, i.e.

$$I(Y(x)=i) = \begin{cases} 1 & \text{if } Y(x) = i \\ 0 & \text{otherwise} \end{cases}$$

(2.4) $$R^k(Y(x)) = \sum_{i=k+1}^{M} b_i I(Y(x)=i) = -\sum_{i=k}^{M-1} \beta_i^k I(Y(x)>i)$$

where $\beta_k^k = -b_{k+1}$ and $\beta_i^k = b_i - b_{i+1}$ for $i = k+1,\dots,M-1$

$$\text{i.e.} \quad b_i = -\sum_{j=k}^{i-1} \beta_j^k \qquad i = k+1,\dots,M .$$

Thus $C^k(Y(x)) = c_i$ and $R^k(Y(x)) = b_i$ if the system is in state i at time x (when k is the control limit).

Note that $Y(x) > i \iff R^i > x$.

3. Multistate age replacement policy (MARP)

We consider here replacement rules of the following form: "Replace at failure (k) or at system age T, whichever occurs first" where $k \in S \subseteq \{0,1,\ldots,M-1\}$. With "$T = \infty$" we mean "Replace at failure (k) only". Unless otherwise specified T is taken to be a constant. If T is a random variable, independently chosen from a fixed distribution $G(\cdot)$ for each scheduled replacement, we call the policy "random multistate age replacement policy" (RMARP). Theorem 3.6 states that we need only consider non-random multistate age replacement policies in seeking the optimum policy if the distributions are continuous.

Let $\tau_i^{T,j} = \min(R_i^j, T)$ for $j = k, k+1, \ldots, M-1$ and $i = 1,2,\ldots$ where R_i^j is the lifelength in the states $G_j^c = \{j+1,\ldots,M\}$ associated with the "i-th system".

Then $\{\tau_i^{T,k}\}_{i=1}^{\infty}$ generates a renewal process. We call $\tau_i^{T,k}$ the i-th cycle. Write $\tau^{T,k}$ for an arbitrary cycle.

We now define $B^{T,k}$ ($B_\alpha^{T,k}$) as the expected average long run cost per unit time (expected total discounted cost) under an MARP when k is the control limit and α is a positive discount factor. From Appendix A.2 it follows that

$$(3.1) \qquad B^{T,k} = \frac{EV^{T,k}}{E\tau^{T,k}}$$

where $V^{T,k}$ is the cost of one cycle.

From Appendix B.1 we see that

$$(3.2) \qquad B_\alpha^{T,k} = \frac{EV_\alpha^{T,k}}{EW_\alpha^{T,k}}$$

where $V_\alpha^{T,k}$ =is the discounted cost of one cycle and $W_\alpha^{T,k} = 1-e^{-\alpha\tau^{T,k}}$.

Our problem is to minimize $B^{T,k}(B_\alpha^{T,k})$ with respect to T for each $k \in S$.

Assume T_k minimizes $B^{T,k}(B_\alpha^{T,k})$. The optimal MARP describes a k^* and a T_{k^*} such that

$$B^{T^*,k^*} = \min_{k \in S}\{B^{T_k,k}\} \quad (B^{T^*,k^*} = \min_{k \in S}\{B_\alpha^{T_k,k}\}) .$$

Before we give explicit expressions for $B^{T,k}$ and $B_\alpha^{T,k}$ we need a lemma.

<u>Lemma 3.1</u>

For $j = 0,1,\ldots,M-1$ and $\alpha > 0$

i) $$E\tau^{T,j} = \int_0^T \bar{F}^j(t)dt$$

ii) $$E\{1-e^{-\alpha\tau^{T,j}}\} = \alpha \int_0^T \bar{F}^j(t)e^{-\alpha t}dt$$

<u>Proof</u>

i) Trivial.

ii) Let $H^j(\cdot)$ be the distribution function for $\tau^{T,j} = \min(T,R^j)$

Then clearly $$\bar{H}^j(t) = \begin{cases} \bar{F}(t) & \text{if} \quad t < T \\ 0 & \text{if} \quad t \geq T \end{cases}$$

Using Appendix B.2 we see that

$$E\{1-e^{-\alpha\tau^{T,j}}\} = 1-\int_0^\infty e^{-\alpha s}dH^j(s) = 1-\alpha \int_0^\infty e^{-\alpha s}H^j(s)ds$$

$$= \alpha \int_0^\infty e^{-\alpha s}ds - \alpha \int_0^\infty e^{-\alpha s}H^j(s)ds = \alpha \int_0^\infty e^{-\alpha s}\bar{H}^j(s)ds$$

$$= \alpha \int_0^T e^{-\alpha s}\bar{F}^j(s)ds .$$

Theorem 3.2

For $T \geq 0$ (included $T = \infty$), $\alpha > 0$ and $k \in S$:

$$(3.3) \quad B^{T,k} = \frac{1}{\int_0^T \bar{F}^k(x)dx}\left[-\sum_{j=k}^{M-1} \beta_j^k \int_0^T \bar{F}^j(x)dx + \sum_{i=k}^{M-1} \gamma_i^k F^i(T) + c_M + \sum_{i=0}^{k} c_i Q_i^k(T)\right]$$

$$(3.4) \quad B_\alpha^{T,k} = \frac{1}{\alpha\int_0^T e^{-\alpha u}\bar{F}^k(u)du}\left[-\sum_{j=k}^{M-1} \beta_j^k \int_0^T e^{-\alpha u}\bar{F}^j(u)du + e^{-\alpha T}\left(\sum_{i=k}^{M-1} \gamma_i^k F^i(T) + c_M\right) + \sum_{i=0}^{k} c_i \int_0^T e^{-\alpha t} dQ_i^k(t)\right]$$

Proof

We see that the numerator in (3.1) is the numerator in (3.2) with $\alpha = 0$. Hence we have to find the expected discounted cost in one cycle of length $\tau^{T,k}$.

Since $R^k(Y(x)) = b_i$ if the system is in the state i at x and $C^k(Y(\tau^{T,k})) = c_i$ if the system is replaced from state i at $\tau^{T,k}$, it follows that

$$EV_\alpha^{T,k} = E\left\{\int_0^{\tau^{T,k}} R^k(Y(x))e^{-\alpha x}dx + C^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}}\right\} .$$

Now using that $R^k(Y(x)) = 0$ for $T \geq x > \tau^{T,k}$ and writing $R^k(Y(\cdot))$ according to (2.4) we get

$$E\left\{\int_0^{\tau^{T,k}} R^k(Y(x))e^{-\alpha x}dx\right\} = E\left\{\int_0^T R^k(Y(x))e^{-\alpha x}dx\right\}$$

$$= -\sum_{i=k}^{M-1} \beta_i^k E \int_0^T I(R^i > x)e^{-\alpha x}dx = -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx .$$

We now write
$$EC^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}} = EC_1^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}} + EC_2^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}}$$

according to (2.1), (2.2) and (2.3).

Conditioning on the time for failure (k) and the first state in G_k we may write

$$EC_1^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}} = \sum_{i=1}^{k}\int_0^T c_i e^{-\alpha t}dQ_i^k(t) \quad \text{since}$$

the expected discounted cost of a replacement is $c_i e^{-\alpha t}$ if the system fails (k) at $t \leq T$ and the first state in G_k is i.

Since $I(Y(\tau^{T,k})=i)e^{-\alpha\tau^{T,k}} = I(Y(T)=i)e^{-\alpha T}$ if $i \geq k+1$,

$$EC_2^k(Y(\tau^{T,k}))e^{-\alpha\tau^{T,k}} = -\sum_{i=k}^{M-1}\gamma_i^k EI(R^i>T)e^{-\alpha T}$$

$$= -\sum_{i=k}^{M-1}\gamma_i^k\bar{F}^i(T)e^{-\alpha T} = \left(\sum_{i=k}^{M-1}\gamma_i^k F^i(T) + c_M\right)e^{-\alpha T}$$

The expressions for the denominators follow from Lemma 3.1, and the proof is complete.

Corollary 3.3

$$(3.5) \qquad \lim_{T\to\infty} B^{T,k} = B^{\infty,k} = \frac{1}{r_k}\left[-\sum_{i=k}^{M-1}\beta_i^k r_i + \sum_{i=0}^{k}c_i P_i^k\right]$$

$$(3.6) \qquad \lim_{T\to\infty} B_\alpha^{T,k} = B_\alpha^{\infty,k} = \frac{1}{\alpha\int_0^\infty e^{-\alpha u}\bar{F}^k(u)du}\left[-\sum_{i=k}^{M-1}\beta_i^k\int_0^\infty e^{-\alpha u}\bar{F}^i(u)du + \sum_{i=0}^{k}c_i\int_0^\infty e^{-\alpha t}dQ_i^k(t)\right]$$

Proof

Let $T = \infty$ in (3.3) and (3.4).

Observe that $\int_0^\infty e^{-\alpha u}\bar{F}^i(u)du$ is the Laplace transform of $\bar{F}^i(\cdot)$.

Remark 3.4

There is an alternative way to find the expected cost due to $b_{k+1},\ldots,b_M$ in one cycle.

Clearly this cost equals $\sum_{i=k+1}^{M} b_i$ · expected time spent in state i in one cycle =

$$\sum_{i=k+1}^{M} b_i E[\tau^{T,i-1} - \tau^{T,i}]$$

$$= - \sum_{i=k}^{M-1} \beta_i^k E\tau^{T,i}$$

where $\tau^{T,M} \equiv 0$.

Remark 3.5

The first integral in $B^{T,k}$ and $B_\alpha^{T,k}$ equals the denominator. Since $\beta_i^k = b_i - b_{i+1}$ for $i \geq k+1$, we infer that it is only the differences between the b_i's that are important for our minimazation problem. (See page 10 and 11.)

Furthermore it follows by Lebesgue Convergence Theorem that $\lim_{\alpha \to 0} \alpha B_\alpha^{T,k} = B^{T,k}$.

We now show that when seeking for the optimal RMARP it is no restriction to assume T is fixed if the distributions $Q_i^j(\cdot)$ are continuous.

Theorem 3.6

Assume all $Q_i^j(\cdot)$'s are continuous. Then the optimal random multistate age replacement policy (RMARP) is non-random.

Proof

Assume T is chosen from the distribution $G(\cdot)$.

Then

$$B^{T,k} = \frac{EV^{T,k}}{E\tau^{T,k}} = \frac{\int_0^\infty E\{V^{T,k} | T=x\} dG(x)}{\int_0^\infty E\{\tau^{T,k} | T=x\} dG(x)} = \frac{\int_0^\infty EV^{x,k} dG(x)}{\int_0^\infty E\tau^{x,k} dG(x)}$$

From (3.3) we see that $B^{x,k} = \frac{EV^{x,k}}{E\tau^{x,k}}$ (x fixed) is continuous in x if $Q_i^j(\cdot)\ \forall i,j$ is continuous. Hence $B^{x,k}$ has a minimum in x, say x_0 (including $x_0 = \infty$). Then

$$
\begin{gathered}
B^{x,k} \geq B^{x_0,k} \qquad \forall x \\
\Downarrow \\
EV^{x,k} \geq B^{x_0,k} E\tau^{x,k} \\
\Downarrow \\
\int_0^\infty EV^{x,k} dG(x) \geq B^{x_0,k} \int_0^\infty E\tau^{x,k} dG(x) \\
\Downarrow \\
B^{T,k} = \frac{\int_0^\infty EV^{x,k} dG(x)}{\int_0^\infty E\tau^{x,k} dG(x)} \geq B^{x_0,k}
\end{gathered}
$$

Q.E.D.

This theorem has been shown by Karlin, cf. Barlow and Prochan (1965) page 86-87.

We have shown the proof for the expected average cost case, it is clear that the proof for the expected total discounted cost case is similar.

We will now study the functions $B^{x,k}$ and $B_\alpha^{x,k}$ as functions of x. In particular we are interested in knowing when preventive replacements are advantageous, i.e. when minimum is obtained for a finite x.

We see from (3.3) and (3.4) that $\lim_{x \to 0^+} B^{x,k} = \infty$, the costs for a small x is of course enormous since we replace the system "almost continuously".

As noted in the proof of Theorem 3.6 $B^{x,k}(B_\alpha^{x,k})$ has a minimum (included $x = \infty$) if the distributions $Q_i^j(\cdot)\ \forall i,j$ are continuous.

We will now assume absolutely continuous distributions.

Minimizing the expected average long run cost per unit time, $B^{x,k}$

Before we state conditions when the minimum of $B^{x,k}$ is obtained for finite x or not, we need a definition.

Let
$$a^k(x) = -\sum_{i=k}^{M-1} \beta_i \frac{{}^k\bar{F}^i(x)}{\bar{F}^k(x)} + \sum_{i=0}^{k} c_i \frac{q_i^k(x)}{\bar{F}^k(x)} + \sum_{i=k}^{M-1} \gamma_i \frac{{}^k f^i(x)}{\bar{F}^k(x)}$$

$$= -\sum_{i=k}^{M-1} \beta_i \frac{{}^k\bar{F}^i(x)}{\bar{F}^k(x)} + \left(\sum_{i=0}^{k} c_i g_i^k(x) - c_{k+1}\right) v^k(x) + \sum_{i=k+1}^{M-1} \gamma_i \frac{{}^k f^i(x)}{\bar{F}^k(x)}$$

$$= \frac{1}{\bar{F}^k(x)} \frac{d}{dx} EV^{x,k}$$

where $v^k(x) = \dfrac{f^k(x)}{\bar{F}^k(x)}$, the failure (k) rate.

It follows that

$$EV^{x,k} = \int_0^x a^k(t)\bar{F}^k(t)dt + EV^{0,k} = \int_0^x a^k(t)\bar{F}^k(t)dt + c_M \tag{3.7}$$

Note that $a^k(x)$ is defined for those x such that $\bar{F}^k(x) > 0$ only, however, this makes no problem since we are not interested in x's such that $\bar{F}^k(x) = 0$.

We will now assume $\lim_{x\to\infty} a^k(x)$ exists.

Theorem 3.7

i) If the equation $B^{x,k} = a^k(x)$ has no finite solution, then the minimum of $B^{\cdot,k}$ is obtained for $x = \infty$.

ii) If $\lim_{x\to\infty} a^k(x) > B^{\infty,k}$, then there exists a finite x_0 minimizing $B^{\cdot,k}$.

iii) If $a^k(x)$ is strictly increasing and $\lim_{x\to\infty} a^k(x) > B^{\infty,k}$, then there exists a unique, finite x_0 minimizing $B^{\cdot,k}$.

iv) If $\lim_{x\to\infty} a^k(x) < B^{\infty,k}$ and $a^k(x)$ is increasing, then $x = \infty$ minimizes $B^{\cdot,k}$, and $B^{x,k}$ is strictly decreasing in x.

Proof

A finite optimal solution (i.e. a x_0 minimizing $B^{\cdot,k}$) must satisfy $\frac{d}{dx} B^{x,k} = 0$. A straightforward calculation yields:

$$\begin{aligned} &\frac{d}{dx} B^{x,k} = 0 \Leftrightarrow B^{x,k} = a^k(x) \\ (3.8) \qquad &\frac{d}{dx} B^{x,k} > 0 \Leftrightarrow B^{x,k} < a^k(x) \\ &\frac{d}{dx} B^{x,k} < 0 \Leftrightarrow B^{x,k} > a^k(x) \end{aligned}$$

It follows that $a^k(x)$ and $B^{x,k}$ intersect at all extremum points of $B^{x,k}$ so that $a^k(x)$ crosses from below (above) at the minima (maxima).

i) Already proved.

ii) If $\lim_{x\to\infty} a^k(x) > B^{\infty,k}$, then there exists an N such that

$$B^{x,k} < a^k(x) \quad \text{for} \quad x \geq N. \quad \text{Hence} \quad \frac{d}{dx} B^{x,k} > 0 \quad \text{for} \quad x \geq N,$$

and it follows that $B^{x,k}$ converges to $B^{\infty,k}$ from below. Clearly there exists a finite x_0 minimizing $B^{\cdot,k}$ in this case.

iii) By ii) there exists an optimal solution. We must show that this is unique.
Now if $a^k(x)$ had intersected $B^{x,k}$ more than once, it must have been at a maximum, and hence $B^{x,k}$ is crossed by $a^k(x)$ from above at such a point, a contradiction to the assumption that $a^k(x)$ is strictly increasing. See Figure 3.1.

iv) If $\lim_{x\to\infty} a^k(x) < B^{\infty,k}$ and $a^k(x)$ is increasing, we infer that $a^k(x) < B^{x,k}$ $\forall x$. See Figure 3.2. By (3.8) it follows that $\frac{d}{dx} B^{x,k} < 0$ $\forall x$.
Thus the minimum is obtained for $x = \infty$, and $B^{x,k}$ is strictly decreasing in x.

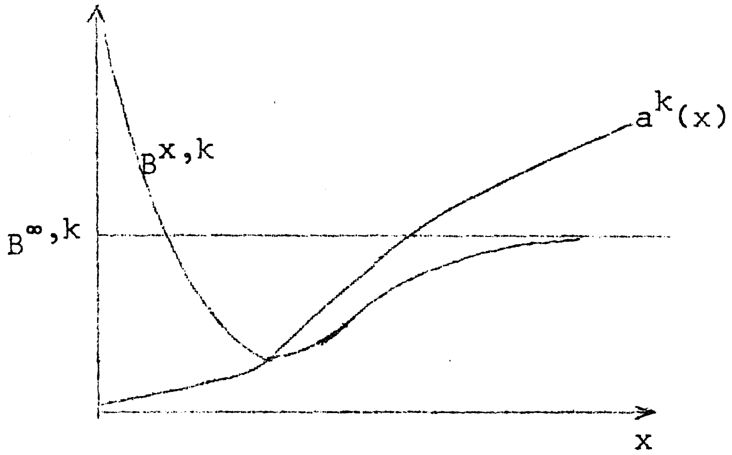

Fig. 3.1 Example of case iii)

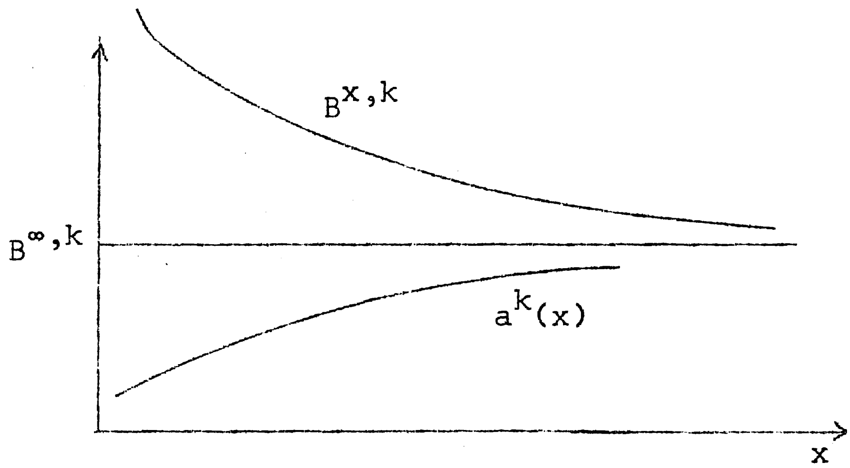

Fig.3.2. Example of case iv)

<u>Remark 3.8</u>

If $c_M = c_{M-1} = \ldots = c_{k+1}$, F^k is IFR*, $v^i(x) \geq v^k(x)\ \forall x$ for $i \geq k+1$ and $\sum_{i=0}^{k} c_i g_i^k(x)$ is increasing, then $a^k(x)$ is increasing.

To see this we write

$$a^k(x) = b_{k+1} - \sum_{i=k+1}^{M-1} \beta_i^k e^{-[\int_0^x (v^i(t)-v^k(t))dt]} + \left(\sum_{i=0}^{k} c_i g_i^k(x) - c_{k+1}\right) v^k(x)$$

* i.e. F^k has an increasing failure rate.

and note that $\beta_i^k \geq 0$ for $i \geq k+1$, $\sum_{i=0}^{k} c_i g_i^k(x) - c_{k+1} \geq c_k - c_{k+1} \geq 0$ and $v^k(x)$ is increasing since F^k is IFR.

The assumption that $\sum_{i=0}^{k} c_i g_i^k(x)$ is increasing in x states roughly that the probability to move to the worst states when we fail (k) at x is increasing in x.

Remark 3.9

If $M = 1$ and $k = 0$ ($b_1 = 0$) we have the binary case with

$$B^{x,0} = \frac{1}{\int_0^x \bar{F}^0(t)dt} [(c_0 - c_1)F^0(x) + c_1] \quad \text{and}$$

$$a^0(x) = (c_0 - c_1)v^0(x) .$$

We now want to give an economic interpretation of Theorem 3.7 by use of marginal cost analysis.

Marginal cost analysis for MARP

Berg (1980) defined the notation of the marginal cost of a preventive replacement in the binary case. The resulting function was shown to be a useful tool for an optimality analysis. The use of marginal cost notation makes the mathematical results for the model meaningful.

We now define the marginal cost of an age replacement in the multistate case.

The marginal cost of an age replacement in the multistate case $= \lim_{\Delta \to 0} \frac{G(\Delta)}{\Delta}$, where

$G(\Delta)$ = Expected cost associated with waiting an additional short time Δ - Expected cost of a replacement now at x.

More precisely $G(\Delta) = E\{(\text{cost in } (x,\tau^{x+\Delta,k}] - \text{cost of an replacement now at } x) | R^k > x\}$. Since the expected cost in $[0,x)$ when replacing at $\tau^{x+\Delta,k}$ given $R^k > x$ equals the expected cost in $[0,x)$ when replacing at $\tau^{x,k}$ given $R^k > x$, $G(\Delta) = E\{(V^{x+\Delta,k} - V^{x,k}) | R^k > x\}$ where $V^{x,k}$ is the cost of one cycle of length $\tau^{x,k}$.

Note that we carry out a replacement if the system fails (k) in $(x,x+\Delta)$, whereas if no such failure (k) occurs a preventive replacement will take place at the end of the interval $(x,x+\Delta)$.

Theorem 3.10

The marginal cost of an age replacement at x is $a^k(x)$.

Proof

From Theorem 3.2 and the proof of this theorem we see that

$$\frac{G(\Delta)}{\Delta} = \frac{1}{\Delta}\Big\{-\sum_{j=k}^{M-1}\beta_j^k \int_x^{x+\Delta} P\{R^j > u | R^k > x\}du - \sum_{i=k}^{M-1}\gamma_i^k [P\{R^i > x+\Delta | R^k > x\} - P\{R^i > x | R^k > x\}]$$

$$+ \sum_{i=0}^{k} c_i [P\{R^k \leqq x+\Delta, \theta^k = i | R^k > x\} - P\{R^k \leqq x, \theta^k = i | R^k > x\}]\Big\}$$

$$= \frac{1}{\Delta}\Big\{-\sum_{j=k}^{M-1}\beta_j^k \int_x^{x+\Delta} \bar{F}^j(u)\,du - \sum_{i=k}^{M-1}\gamma_i^k[\bar{F}^i(x+\Delta) - \bar{F}^i(x)] + \sum_{i=0}^{k} c_i [Q_i^k(x+\Delta) - Q_i^k(x)]\Big\} \frac{1}{\bar{F}^k(x)} .$$

This implies that $\lim_{\Delta\to 0^+} \frac{G(\Delta)}{\Delta} = \frac{1}{\bar{F}^k(x)} \frac{d}{dx} EV^{x,k} = a^k(x)$.

This interpretation of $a^k(x)$ help us to better understand the conditions in Theorem 3.7, for example the assumption that $a^k(x)$ is strictly increasing and $\lim_{x\to\infty} a^k(x) < B^{\infty,k}$. Then the marginal cost of an age replacement is always less than average cost (see Th. 3.7 iv) and preventive replacements are unwarranted since for all preventive replacement times x, it cost less in average to increase x.

Note that there is a similarity between the marginal cost analysis and the infinitesimal-look-ahead-gain in certain Markov models. See Ross (1971).

Minimizing the expected total discounted cost function $B_\alpha^{x,k}$

Since minimizing $B_\alpha^{x,k}$ is similar to minimizing $B^{x,k}$, we will just list the main definitions and results.

Let
$$a_\alpha^k(x) = -\sum_{i=k}^{M-1} \beta_i^k \frac{\bar{F}^i(x)}{\bar{F}^k(x)} + \sum_{i=0}^{k} c_i \frac{q_i^k(x)}{\bar{F}^k(x)} + \sum_{i=k}^{M-1} \gamma_i^k \frac{f^i(x)}{\bar{F}^k(x)} - \alpha \sum_{i=k}^{M-1} \gamma_i^k \frac{F^i(x)}{\bar{F}^k(x)} - \frac{\alpha c_M}{\bar{F}^k(x)}$$
$$= \frac{e^{\alpha x}}{\bar{F}^k(x)} \frac{d}{dx} EV_\alpha^{x,k}.$$
Note that $-\left\{\sum_{i=k}^{M-1} \gamma_i^k F^i(x) + c_M\right\} = \sum_{i=k}^{M-1} \gamma_i^k \bar{F}^i(x)$.

Clearly $\lim_{\alpha\to 0^+} a_\alpha^k(x) = a^k(x)$. Similarly to (3.8) we get

$$\frac{d}{dx} B_\alpha^{x,k} = 0 \iff \alpha B_\alpha^{x,k} = a_\alpha^k(x)$$
$$\frac{d}{dx} B_\alpha^{x,k} > 0 \iff \alpha B_\alpha^{x,k} < a_\alpha^k(x)$$
$$\frac{d}{dx} B_\alpha^{x,k} < 0 \iff \alpha B_\alpha^{x,k} > a_\alpha^k(x)$$

It follows that $a_\alpha^k(x)$ and $\alpha B_\alpha^{x,k}$ intersect at all extremum points of $B_\alpha^{x,k}$ so that $a_\alpha^k(x)$ crosses from below (above) at the minima (maxima).

Theorem 3.11

i) If the equation $\alpha B_\alpha^{x,k} = a_\alpha^k(x)$ has no finite solution, then the minimum of $B_\alpha^{\cdot,k}$ is obtained for $x = \infty$.

ii) If $\lim_{x\to\infty} a_\alpha^k(x) > \alpha B_\alpha^{\infty,k}$, then there exists a finite x_0 minimizing $B_\alpha^{\cdot,k}$.

iii) If $\lim\limits_{x\to\infty} a_\alpha^k(x) > \alpha B_\alpha^{\infty,k}$ and $a_\alpha^k(x)$ is strictly increasing, then there exists a unique finite x_0 minimizing $B_\alpha^{\cdot,k}$.

iv) If $\lim\limits_{x\to\infty} a_\alpha^k(x) < \alpha B_\alpha^{\infty,k}$ and $a_\alpha^k(x)$ is increasing, then $x = \infty$ minimizes $B_\alpha^{\cdot,k}$ and $B_\alpha^{x,k}$ is strictly increasing in x.

<u>Proof</u>

See Theorem 3.7.

<u>Theorem 3.12</u>

The marginal cost of an age replacement at x in the discounted case is $a_\alpha^k(x)$.

<u>Proof</u> Let

$G_\alpha(\Delta)$ = Expected discounted cost associated with waiting an additional short time Δ - Expected cost of an age replacement now at x.

Now

$$G_\alpha(\Delta) = E\{e^{\alpha x}[V_\alpha^{x+\Delta,k} - V_\alpha^{x,k}] \mid R^k > x\} \quad \text{(see page 26)},$$

the factor $e^{\alpha x}$ is caused by the fact that all costs are discounted to time x.

From Theorem 3.2 and the proof of this theorem we see that

$$\frac{G_\alpha(\Delta)}{\Delta} = e^{\alpha x}\frac{1}{\Delta}\Bigg\{-\sum_{j=k}^{M-1}\beta_j^k\int_x^{x+\Delta}\frac{\bar F^j(u)}{\bar F^k(x)}\,e^{-\alpha u}du$$

$$-\sum_{i=k}^{M-1}\gamma_i^k\,\frac{[\bar F^i(x+\Delta)e^{-\alpha(x+\Delta)}-\bar F^i(x)e^{-\alpha x}]}{\bar F^k(x)}+\sum_{i=0}^{k}c_i\int_x^{x+\Delta}\frac{e^{-\alpha t}}{\bar F^k(x)}\,dQ_i^k(t)\Bigg\}$$

$$= \frac{e^{\alpha x}}{\bar F^k(x)}\,\frac{1}{\Delta}\,(EV_\alpha^{x+\Delta,k} - EV_\alpha^{x,k}) .$$

This gives

$$\lim_{\Delta \to 0^+} \frac{G_\alpha(\Delta)}{\Delta} = \frac{e^{\alpha x}}{\bar{F}^k(x)} \frac{d}{dx} EV_\alpha^{x,k} = a_\alpha^k(x) .$$

4. A general multistate replacement model

Let $X(t) = (X_1(t),\dots,X_n(t))$ be a stochastic vector process. $X(t)$ is an observable stochastic characteristic of the system at time t influencing the deterioration process.

Let $\tilde{Q}_j^k(t) = P\{R^k \leq t,\ \theta^k = j \mid X(s),\ s \geq 0\} = \int_0^t \tilde{q}_j^k(s)ds$ for $j \leq k$.

Similarly we define

$$F^k(t) = P\{R^k \leq t \mid X(s),\ s \geq 0\} = \int_0^t f^k(s)ds \quad \text{and}$$

$$\tilde{a}^k(t) = -\sum_{i=k}^{M-1} \beta_i^k \frac{\tilde{F}^i(t)}{\tilde{\bar{F}}^k(t)} + \sum_{i=0}^{k} c_i \frac{\tilde{q}_i^k(t)}{\tilde{\bar{F}}^k(t)} + \sum_{i=k}^{M-1} \gamma_i^k \frac{\tilde{f}^i(t)}{\tilde{\bar{F}}^k(t)} .$$

We will now assume $\tilde{a}^k(t)$ is right continuous in t and increasing in t almost sure (a.s.). The latter assumption is similar to the assumption of an increasing $a^k(t)$ in MARP.

Furthermore let T be a stopping time such that the event $\{T \leq t\}$ is determined by $\{X(s),\ 0 \leq s \leq t\}$ and an auxiliary randomizing experiment which is independent of "everything".

The class of replacement rules will here be of the form: "Replace at failure (k) or at T, whichever occurs first", $k \in S$. "$T = \infty$" means "Replace at failure (k) only".

The problem is to minimize the expected average long run cost per unit time, $B^{T,k}$. (We use the same notation as in MARP although T now is random.)

From Appendix A.2 it follows that

$$B^{T,k} = \frac{EV^{T,k}}{E\tau^{T,k}}$$

where $\tau^{T,k} = \min(T, R^k)$ and $V^{T,k}$ is the cost in one cycle of length $\tau^{T,k}$.

Conditioning on $X(s)$, $s \geq 0$ and the randomizing experiment we may write

$$B^{T,k} = \frac{EV^{T,k}}{E\tau^{T,k}} = \frac{E\tilde{E}V^{T,k}}{E\tilde{E}\tau^{T,k}}$$

where $\tilde{E}$ means expectation given $X(s)$, $s \geq 0$ and the experiment.

We now infer that $\tilde{E}V^{T,k}$ and $\tilde{E}\tau^{T,k}$ must be identical to $EV^{T,k}$ and $E\tau^{T,k}$ in Theorem 3.2 with ~ notation everywhere since the stopping time T may be regarded as a fixed number for given $X(s)$, $s \geq 0$ and the randomizing experiment.

Thus

$$(4.1) \qquad \tilde{E}V^{T,k} = \int_0^T \tilde{a}^k(t)\tilde{\bar{F}}^k(t)dt + c_M \quad \text{by (3.7)}$$

$$\tilde{E}\tau^{T,k} = \int_0^T \tilde{\bar{F}}^k(t)dt \qquad \text{and}$$

$$B^{T,k} = \frac{E\int_0^T \tilde{a}^k(t)\tilde{\bar{F}}^k(t)dt + c_M}{E\int_0^T \tilde{\bar{F}}^k(t)dt}$$

We see that if $E\tau^{T,k} = 0$ for a stopping time T, then $T = 0$ a.s., and it follows that $B^{T,k} = \infty$.

Before we show the form of the optimal rule, we need two lemmas which are useful far outside our model. (cf. Bergman (1980))

Lemma 4.1

If T_λ is a stopping time which minimizes $B_\lambda^T = EV^T - \lambda E\tau^T$, $(\lambda > 0)$ and $B_\lambda^{T_\lambda} = 0$, then T_λ minimizes $B^T = \dfrac{EV^T}{E\tau^T}$.

Proof

Since $0 = B_\lambda^{T_\lambda} = EV^{T_\lambda} - \lambda E\tau^{T_\lambda} \leq EV^T - \lambda E\tau^T$ for every stopping time T, it follows that

$$\lambda = \frac{EV^{T_\lambda}}{E\tau^{T_\lambda}} \leq \frac{EV^T}{E\tau^T} .$$

Lemma 4.2

Let $\lambda_0 = \inf_T \frac{EV^T}{E\tau^T}$. Assume T_{λ_0} minimizes $B^T_{\lambda_0}$ and $E\tau^T < K$ = constant for every stopping time T. Then $B_{\lambda_0}^{T_{\lambda_0}} = 0$.

Proof

From the definition of λ_0 we see that

$$\lambda_0 \leq \frac{EV^{T_{\lambda_0}}}{E\tau^{T_{\lambda_0}}} .$$

Hence

$$EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \geq 0 .$$

It remains to show that $EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \leq 0$.

Let $\varepsilon > 0$ be given. By the definition of λ_0 there exists a stopping time $T(\varepsilon)$ such that

$$\lambda_0 + \varepsilon > \frac{EV^{T(\varepsilon)}}{E\tau^{T(\varepsilon)}} .$$

This implies

$$EV^{T(\varepsilon)} - \lambda_0 E\tau^{T(\varepsilon)} < \varepsilon E\tau^{T(\varepsilon)} ,$$

and it follows that

$$EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \leq EV^{T(\varepsilon)} - \lambda_0 E\tau^{T(\varepsilon)} < \varepsilon E\tau^{T(\varepsilon)} < \varepsilon K .$$

Since ε was arbitrary, $B_{\lambda_0}^{T_{\lambda_0}} = 0$ and the lemma is proved.

$$\lambda = \frac{EV^{T_\lambda}}{E\tau^{T_\lambda}} \leqq \frac{EV^T}{E\tau^T} .$$

Lemma 4.2

Let $\lambda_0 = \inf_T \frac{EV^T}{E\tau^T}$. Assume T_{λ_0} minimizes $B^T_{\lambda_0}$ and $E\tau^T < K$ = constant for every stopping time T. Then $B_{\lambda_0}^{T_{\lambda_0}} = 0$.

Proof

From the definition of λ_0 we see that

$$\lambda_0 \leqq \frac{EV^{T_{\lambda_0}}}{E\tau^{T_{\lambda_0}}} .$$

Hence

$$EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \geqq 0 .$$

It remains to show that $EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \leqq 0$.

Let $\varepsilon > 0$ be given. By the definition of λ_0 there exists a stopping time $T(\varepsilon)$ such that

$$\lambda_0 + \varepsilon > \frac{EV^{T(\varepsilon)}}{E\tau^{T(\varepsilon)}} .$$

This implies

$$EV^{T(\varepsilon)} - \lambda_0 E\tau^{T(\varepsilon)} < \varepsilon E\tau^{T(\varepsilon)} ,$$

and it follows that

$$EV^{T_{\lambda_0}} - \lambda_0 E\tau^{T_{\lambda_0}} \leqq EV^{T(\varepsilon)} - \lambda_0 E\tau^{T(\varepsilon)} < \varepsilon E\tau^{T(\varepsilon)} < \varepsilon K .$$

Since ε was arbitrary, $B_{\lambda_0}^{T_{\lambda_0}} = 0$ and the lemma is proved.

We are now able to give the form of the optimal rule, (note that $\inf\{\emptyset\} = \infty$) .

Theorem 4.3

Let $T_\lambda = \inf\{t: \tilde{a}^k(t) \geq \lambda\}$ and $\lambda_0 = \inf_T B^{T,k} = \inf_T \frac{EV^{T,k}}{E\tau^{T,k}}$.

Then the stopping time T_{λ_0} minimizes $B^{T,k}$.

Proof

Clearly $E\tau^{T,k} \leq ER^k = r_k < \infty$.

$$\text{Define} \quad B_\lambda^T = E\{c_M + \int_0^T \tilde{a}^k(t)\tilde{\bar{F}}^k(t)dt - \lambda \int_0^T \tilde{\bar{F}}^k(t)dt\}$$

$$= E\{c_M + \int_0^T (\tilde{a}^k(t) - \lambda)\tilde{\bar{F}}^k(t)dt\} .$$

Since $\tilde{a}^k(t)$ is increasing a.s., we see that the integral will be minimized if we "stop" as soon as $\tilde{a}^k(t) - \lambda$ becomes positive. Thus $T_\lambda = \inf\{t: \tilde{a}^k(t) \geq \lambda\}$ minimizes B_λ^T for given $X(s)$, $s \geq 0$ and the experiment, and hence T_λ minimizes B_λ^T unconditionally.

Now by Lemma 4.2 and 4.1 we see that T_{λ_0} minimizes $B^{T,k}$. Q.E.D.

Even though $\lambda_0 = \inf_T B^{T,k} = B^{T_{\lambda_0},k}$ is in general unknown, the theorem is quite useful as it enables us to determine the structure of the optimal rule.

We will now discuss the stopping times T_λ

Define

$$B^k(\lambda) = B^{T_\lambda,k} .$$

It follows that $\lambda_0 = B^k(\lambda_0)$.

Theorem 4.4

If $\lambda > \lambda_0$, then

$$\lambda > B^k(\lambda) .$$

Proof

It is equivalently to prove that $B_\lambda^{T}\lambda = EV^{T_\lambda,k} - \lambda E\tau^{T_\lambda,k} < 0$.
Since T_λ minimizes B_λ^{T} , $\lambda > \lambda_0$ and $B_{\lambda_0}^{T}\lambda_0 = 0$,
it follows that

$$B_\lambda^{T}\lambda \leqq B_\lambda^{T}\lambda_0 < B_{\lambda_0}^{T}\lambda_0 = 0$$

We will now state an algorithm for finding λ_0 .
See Bergman (1980).

Theorem 4.5

Let λ_1 be arbitrary with $E\tau^{T_{\lambda_1},k} > 0$.

Then for $i \geqq 2$

$$\lambda_i = B^k(\lambda_{i-1}) \rightarrow \lambda_0 \text{ as } i \rightarrow \infty$$

See Figure 4.1,page 37.

The following lemma is similar to the basic Lemma 3.1 in Bergman (1978).

Lemma 4.6

Let T_1 and T_2 be two stopping times such that $E\tau^{T_i,k} > 0$, $i = 1,2$. Suppose we have a.s.

a) $\tilde{a}^k(t) \leq B^{T_1,k} \quad \forall t: \quad T_1 \leq t < T_2$

$\tilde{a}^k(t) \geq B^{T_1,k} \quad \forall t: \quad T_2 \leq t < T$.

Then $B^{T_1,k} \geq B^{T_2,k}$.

b) $\tilde{a}^k(t) \geq B^{T_1,k} \quad \forall t: \quad T_1 \leq t < T_2$

$\tilde{a}^k(t) \leq B^{T_1,k} \quad \forall t: \quad T_2 \leq t < T_1$.

Then $B^{T_1,k} \leq B^{T_2,k}$.

Proof

We shall first prove the first part of the lemma. We have

$$B^{T_i,k} = \frac{EV^{T_i,k}}{E\tau^{T_i,k}} \qquad i = 1,2 .$$

Hence $B^{T_1,k}\, E\tau^{T_1,k} - B^{T_2,k} E\tau^{T_2,k} = EV^{T_1,k} - EV^{T_2,k}$, which implies

$$B^{T_1,k}[E\tau^{T_2,k} - E\tau^{T_1,k}] = [B^{T_1,k} - B^{T_2,k}]E\tau^{T_2,k} + EV^{T_2,k} - EV^{T_1,k} .$$

Since $E\tau^{T_2,k} > 0$, we must show that

$$B^{T_1,k}[E\tau^{T_2,k} - E\tau^{T_1,k}] \geq EV^{T_2,k} - EV^{T_1,k} .$$

Conditioning on $X(s),\ s \geq 0$ and the randomizing experiment we get for the left hand side

$$B^{T_1,k}E\{\tilde{E}\tau^{T_2,k} - \tilde{E}\tau^{T_1,k}\}$$

$$= E\{B^{T_1,k}(\tilde{E}\tau^{T_2,k} - \tilde{E}\tau^{T_1,k})\} = E\{B^{T_1,k}(\int_0^{T_2} \tilde{\bar{F}}^k(t)dt - \int_0^{T_1} \tilde{\bar{F}}^k(t)dt\}$$

$$= E\int_{T_1}^{T_2} B^{T_1,k}\tilde{\bar{F}}^k(t)dt \geq E\int_{T_1}^{T_2} \tilde{a}^k(t)\tilde{\bar{F}}^k(t)dt .$$

The last inequality follows from the assumptions.

Now we see from (4.1) that

$$E\int_{T_1}^{T_2}\tilde{a}^k(t)\tilde{\bar{F}}^k(t)dt = E\{\widetilde{EV}^{T_2,k} - \widetilde{EV}^{T_1,k}\} = EV^{T_2,k} - EV^{T_1,k} .$$

Thus a) is proved.

b) We must show that $\overset{*}{B}{}^{T_1,k}[E\tau^{T_2,k} - E\tau^{T_1,k}] \leq EV^{T_2,k} - EV^{T_1,k}$.
Using the same tecnique as in a), this is straightforward.

Theorem 4.7

$B^k(\lambda) = B^{T_\lambda,k}$ is decreasing in λ for $\lambda \leq \lambda_0$ and increasing in λ for $\lambda \geq \lambda_0$.

Proof

Let $\lambda_2 \leq \lambda_1 \leq \lambda_0$. Then $T_{\lambda_2} \leq T_{\lambda_1} \leq T_{\lambda_0}$
Using the definition of T_{λ_0} and the fact that T_{λ_0} is optimal, we get

$$\tilde{a}^k(t) < B^k(\lambda_0) \leq B^k(\lambda_1) \quad \text{for all } t \text{ such that } T_{\lambda_2} \leq t < T_{\lambda_1} \leq T_{\lambda_0} .$$

Hence by Lemma 4.6 b) if $E\tau^{T_{\lambda_2},k} > 0$ and $E\tau^{T_{\lambda_1},k} > 0$

$$B^k(\lambda_1) \leq B^k(\lambda_2)$$

If $E\tau^{T_{\lambda_2},k} = 0$ ($E\tau^{T_{\lambda_1},k} = 0$ and $E\tau^{T_{\lambda_2},k} = 0$), then clearly $B^k(\lambda_1) \leq B^k(\lambda_2)$.

Now let $\lambda_0 \leq \lambda_1 \leq \lambda_2$. Using that $\tilde{a}^k(t)$ is increasing a.s. and Theorem 4.4, it follows a.s. that

$$\tilde{a}^k(t) \geq \lambda_1 \geq B^k(\lambda_1) \quad \text{for all } t \text{ such that } T_{\lambda_0} \leq T_{\lambda_1} \leq t < T_{\lambda_2} .$$

Hence by Lemma 4.6 b)

$B^k(\lambda_1) \leq B^k(\lambda_2)$ and the theorem is proved.

In Figure 4.1 we show the principal features of $B^k(\lambda)$.

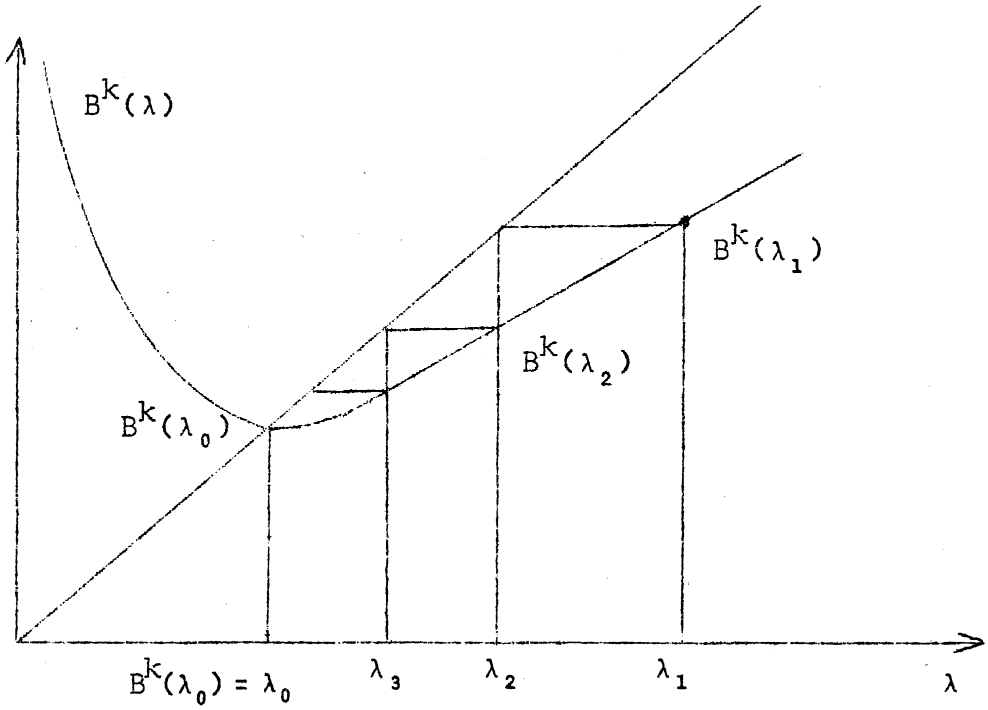

Fig. 4.1

Illustration of the algorithm in Theorem 4.5.

Theorem 4.8

The marginal cost of a replacement at t is $\tilde{a}^k(t)$.

Proof

Let $X(s),\ s \geq 0$ be given and assume $R^k > t$. Then we are in the same situation as in Theorem 3.10 with $\sim$ notation everywhere, and we can copy the proof of Theorem 3.10.

Some examples

1 MARP.

By taking $X(t) \equiv t$ we see that the MARP is a special case of the model described here, however, in this general model we have considered the case when $a^k(t)$ is increasing only.

2 Multistate coherent system

Consider a multistate coherent system of type 2 (an MCS of type 2 - see Natvig (1980)) having n components.

Let $y(t) = (y_1(t),\dots,y_n(t))$ where $y_i(t)$ represents the state of the i-th component at time t.

The state of the system is $\varphi(y(\circ))$. Assume $\varphi(\circ)$ takes values in $\{0,1,\dots,M\}$ whereas the components are binary $\{0,M\}$, independent and have an exponentially distributed time to failure.

Now by the definition of a MCS of type 2 for the special case where components are binary:

$$\varphi(y) \geq j \Leftrightarrow \varphi_j(x) = 1 \quad \forall j \in \{1,\dots,M\}, \quad \forall y,$$

where $\varphi_j(\circ)$ $j = 1,2,\dots,M$ are binary coherent sturctures and

$$x = (x_1,x_2,\dots,x_n), \quad x_i = \begin{cases} 1 & \text{if } y_i = M \\ 0 & \text{if } y_i = 0 \end{cases}$$

Now define for $j = 1,2,\dots,M$

$$v_j(x) = \begin{cases} \sum_{i=1}^{n} \lambda_i x_i (1-\varphi_j(0_i,x))^{*} & \text{if a failure } (j-1) \text{ has not occured } (\varphi_j(x)=1) \\ \infty & \text{if a failure } (j-1) \text{ has occured } (\varphi_j(x)=0) \end{cases}$$

By Bergman (1978) $v_j(X(t))$ is increasing as a function of time. (See Appendix B.3)

Assume $c_1 = c_2 = \dots = c_M$, $S = \{0\}$ and

$\tilde{F}^{j-1}(t) = e^{-\int_0^t v_j(X(s))ds}$. Then

* Notation $(\circ_i,x) = (x_1,x_2,\dots,x_{i-1},\circ,x_{i+1},\dots,x_n)$

$$\tilde{a}^0(t) = -\sum_{j=0}^{M-1} \beta_j^0 \, e^{-[\int_0^t v_{j+1}(X(s))ds - \int_0^t v_1(X(s))ds]} + (c_0 - c_1)v_1(X(t)) .$$

From the definition of $v_j(\cdot)$ we see that $v_j(\cdot) \geq v_1(\cdot) \quad j > 1$. (See Theorem 4.2 Natvig (1980))

Thus $\tilde{a}^0(t)$ is increasing, and the form of the optimal rule is found.

Using the notation of Nummelin (1978) we can formalize the model.

Let (Ω, Σ, P) be a probability space and $\{\mathcal{F}_t\}$ an increasing family of sub-σ-fields of Σ. The σ-field $\mathcal{F}_t$ represents the wear of the system, the accumulated stress or accumulated damage. We suppose at any time $\mathcal{F}_t$ is observable.

Let $\mathcal{F}$ be the σ-field generated by $\mathcal{F}_t$, $\forall t$, i.e.

$$\mathcal{F} = \bigvee_{t \geq 0} \mathcal{F}_t$$

($\mathcal{A}$ and $\mathcal{B}$ sub-σ-fields of Σ, then $\mathcal{A} \vee \mathcal{B}$ is the smallest sub-σ-field of Σ which contains both $\mathcal{A}$ and $\mathcal{B}$.)

Assume $P\{R^k \leq t, \theta^k = i \mid \mathcal{F}\} = P\{R^k \leq t, \theta^k = i \mid \mathcal{F}_t\}$

$$= H_i^k(t) = \int_0^t h_i^k(s)ds \qquad i \leq k , \quad \text{where}$$

$\{h_i^k(t), t \geq 0\}$ is a measurable stochastic process adapted to $\{\mathcal{F}_t\}$.

Let $H^k(t) = P\{R^k \leq t \mid \mathcal{F}_t\}$

and $h^k(t) = \frac{d}{dt} H^k(t)$

Furthermore, let $\mathcal{E}$ be a sub-σ-field of Σ independent of $\mathcal{F}$ and R^j, θ^i $\forall i,j$. We shall denote by $\mathcal{T}$ the set of extended non-negative random variables measurable with respect to $\mathcal{F} \vee \mathcal{E}$. $\mathcal{E}$ can be thought to represent a randomizing experiment independent of everything else. The set $\mathcal{T}$ includes all stopping times and randomized stopping times (rel. to $\{\mathcal{F}_t\}$), which are "randomized by $\mathcal{E}$".

As before we shall consider stopping rules δ_T, $T \in \mathcal{T}$, meaning that the system is replaced at $\min(T,R^k)$.
Note that we may set $\mathcal{F}_t = \sigma(X(s),\ 0 \leq s \leq t)$ if there is a process $X(t)$ describing the wear etc.

If we proceed as before, we find the expected average long run cost per unit time, $B^{T,k}$.

$$B^{T,k} = \frac{E\int_0^T S^k(t)dt + c_M}{E\int_0^T \bar{H}^k(t)dt}$$

where $S^k(t) = -\sum_{i=k}^{M-1} \beta_i^k\, \bar{H}^i(t) + \sum_{i=k}^{M-1} \gamma_i^k\, h^i(t) + \sum_{i=0}^{k} c_i\, h_i^k(t)$

We have used that $E\tau^{T,j} = E\,E\min(T,R^j \mid \mathcal{F} \vee \mathcal{E}) = E\int_0^T \bar{H}^j(t)dt$ etc.

Assume $\alpha^k(t) = \dfrac{S^k(t)}{\bar{H}^k(t)}$ is increasing a.s. and right continuous.

Then $T_{\lambda_0} = \inf\{t: \alpha^k(t) \geq \lambda_0\}$ where $\lambda_0 = \inf_T B^{T,k}$ is optimal.

We can now proceed as before.

5. Multistate block replacement policy (MBRP)

Here we consider replacement rules of the form:
"Replace at failure (k) and at times nT, $n = 1,2,\ldots$ "
$k \in S \subseteq \{0,1,\ldots,M-1\}$.
With "$T = \infty$" we mean the rule "Replace at failure (k) only".
T is a constant.

We write $\bar{B}^{T,k}(\bar{B}^{T,k}_{\alpha})$ for the expected average long run cost per unit time (the total expected, discounted cost) under an MBRP.

Let $w^{T,k}_{\alpha}$ be the expected cost in a block interval of length T, $0 \le T < \infty$, $\alpha \geq 0$.
Then Lemma 5.1 reduces the problem of finding $\bar{B}^{T,k}(\bar{B}^{T,k}_{\alpha})$ to finding $w^{T,k}_{\alpha}$ for $\alpha \geq 0$. This result is not surprising since the process restarts at nT, $n = 1,2,\ldots$.

Lemma 5.1

(i) $\bar{B}^{T,k} = \dfrac{w^{T,k}_{0}}{T}$ (ii) $\bar{B}^{T,k}_{\alpha} = \dfrac{w^{T,k}_{\alpha}}{1-e^{-\alpha T}}$ for $\alpha > 0$.

Proof

(i) Let $nT \le t \le (n+1)T$. Then $n w^{T,k}_{0} \le w^{T,k}_{0}(t) \le (n+1)w^{T,k}_{0}$ where $w^{T,k}_{0}(t)$ is the total expected cost in $[0,t]$ under an MBRP.

Since we obviously have

$$\frac{n}{n+1}\,\frac{w^{T,k}_{0}}{T} \le \frac{w^{T,k}_{0}(t)}{t} \le \frac{n+1}{n}\,\frac{w^{T,k}_{0}}{T},$$

it follows that $\dfrac{w^{T,k}_{0}(t)}{t} \xrightarrow[t\to\infty]{} \dfrac{w^{T,k}_{0}}{T}$. Q.E.D.

(ii) See Appendix B.1.

Since the MBRP is of the form: "Replace at failure (k) only" in the block intervals $(nT,(n+1)T)$ and perform preventive replacements at nT $n=1,2,\ldots$, let us define a right-continuous stochastic process $\{Z(t),\ t \geq 0\}$ representing the state of the system under the policy "Replace at failure (k) only".

Then define for $j = 0,1,\ldots,M-1$

$D_j(s) = P\{Z(s) > j\}$ and

$A_j^k(s) = \int_0^s D_j^k(u)du$ = expected time the system is in the states $\{j+1,\ldots,M\}$ in $[0,s]$.

Note that $P\{Z(T) > j\} = D_j^k(T)$ means the probability that the system will be in the states $\{j+1,\ldots,M\}$ just before we perform the planned replacement at T.

Furthermore let

$M^k(t) = EN^k(t)$ = expected number of failure (k)'s in $[0,t]$ under the policy: "Replace at failure (k) only".

$M_i^k(t) = EN_i^k(t)$ = expected number of replacements from state i $(i \leq k)$ in $[0,t]$ under the policy: "Replace at failure (k) only".

See Appendix A.6 for some properties of $M^k(t)$, the renewal function.

<u>Theorem 5.2</u>

For $T < \infty$, $k \in S$ and $\alpha > 0$

$$\bar{B}^{T,k} = \frac{1}{T}\left[-\sum_{i=k}^{M-1}\beta_i^k A_i^k(T) + \sum_{i=0}^{k} c_i M_i^k(T) - \sum_{i=k}^{M-1}\gamma_i^k D_i^k(T)\right]$$

$$= \frac{1}{T}\left[-\sum_{i=k}^{M-1}\beta_i^k\int_0^T \bar{F}^i(x)(1+M^k(T-x))dx + \sum_{i=0}^{k} c_i\int_0^T(1+M^k(T-u))dQ_i^k(u)\right.$$
$$\left. - \sum_{i=k}^{M-1}\gamma_i^k(\bar{F}^i(T)+\bar{F}^i * M^k(T))\right]$$

$$\bar{B}_\alpha^{T,k} = \frac{1}{1-e^{-\alpha T}}\left[-\sum_{i=k}^{M-1}\beta_i^k\int_0^T e^{-\alpha x}D_i^k(x)dx + \sum_{i=0}^{k}c_i\int_0^T e^{-\alpha s}dM_i^k(s) - e^{-\alpha T}\sum_{i=k}^{M-1}\gamma_i^k D_i^k(T)\right]$$

$$= \frac{1}{1-e^{-\alpha T}}\left[-\sum_{i=k}^{M-1}\beta_i^k\int_0^T e^{-\alpha x}\bar{F}^i(x)[1+\int_0^{T-x}e^{-\alpha u}dM^k(u)]dx\right.$$

$$+\sum_{i=0}^{k}c_i\int_0^T e^{-\alpha x}[1+\int_0^{T-x}e^{-\alpha u}dM^k(u)]dQ_i^k(x)$$

$$\left.-e^{-\alpha T}\sum_{i=k}^{M-1}\gamma_i^k(\bar{F}^i(T)+\bar{F}^i * M^k(T))\right]$$

where

$$D_i^k(s) = \bar{F}^i(s) + \bar{F}^i * M^k(s) = \bar{F}^i(s) + \int_0^s \bar{F}^i(s-x)dM^k(x) \quad s \le T,\ i \ge k .$$

$$A_i^k(T) = \int_0^T \bar{F}^i(x)(1+M^k(T-x))dx\ , \quad i \ge k .$$

$$M_i^k(T) = \int_0^T (1+M^k(T-u))dQ_i^k(u)\ , \quad i \le k .$$

Proof

From Lemma 5.1 we see that it remains to find $w_\alpha^{T,k}$ for $\alpha \ge 0$. Let $w_\alpha^{T,k} = Eb_\alpha^{T,k} + EC_\alpha^{T,k} + ED_\alpha^{T,k}$ where

$b_\alpha^{T,k}$ = discounted cost in $[0,T]$ due to the b_i's , $i \ge k+1$

$C_\alpha^{T,k}$ = discounted cost in $[0,T]$ due to the c_i's , $i \le k$

$D_\alpha^{T,k}$ = discounted cost in $[0,T]$ due to the c_i's , $i \ge k+1$.

Using (2.3) we find

$$ED_\alpha^{T,k} = E\sum_{i=k+1}^{M} c_i I(Z(T)=i)e^{-\alpha T} = -\sum_{i=k}^{M-1}\gamma_i^k D_i^k(T)e^{-\alpha T} .$$

Thus we must find a) $D_i^k(T)$ b) $EC_\alpha^{T,k}$ and c) $Eb_\alpha^{T,k}$.

a) Conditioning on the time for the first failure (k), we get for $i \ge k$

$$D_i^k(s) = P\{Z(s)>i\} = \int_0^\infty P\{Z(s)>i \mid R_1^k=t\}dF^k(t) .$$

Since the process restarts at t, it follows that

$$P\{Z(s)>i \mid R_1^k=t\} = \begin{cases} D_i^k(s-t) & \text{if } s \geq t \\ P\{R_1^i>s \mid R_1^k=t\} & \text{if } s < t \end{cases}$$

Hence $D_i^k(s) = \int_0^s D_i^k(s-t)dF^k(t) + \int_s^\infty P\{R_1^i>s \mid R_1^k=t\}dF^k(t)$

$$= \int_0^s D_i^k(s-t)dF^k(t) + \bar{F}^i(s)$$

Using Appendix A.3,

$$D_i^k(s) = \bar{F}^i(s) + \int_0^s \bar{F}^i(s-t)dM^k(t)$$

b) Conditioning on the time for the first failure (k) and the failure (k) state (i.e. the first state in G_k), we get

$$EC^{T,k} = \sum_{i=0}^{k} \int_0^T E\{C_\alpha^{T,k} \mid R_1^k=t,\ \theta^k=i\}dQ_i^k(t)$$

$$= \sum_{i=0}^{k} c_i \int_0^T e^{-\alpha t}dQ_i^k(t) + \sum_{i=0}^{k} \int_0^T e^{-\alpha t}(EC_\alpha^{T-t,k})dQ_i^k(t)$$

The last equality follows since $EC_\alpha^{T,k}$ equals $(c_i+EC^{T-t,k})e^{-\alpha t}$ given the first failure (k) is at time t, $t \leq T$ and the failure (k) state is i. Since a cost c incurred at time t is equivalent with a cost $ce^{-\alpha t}$ at time 0, we have to multiply $c_i + EC_\alpha^{T-t,k}$ by $e^{-\alpha t}$.

Obviously $E\{C_\alpha^{T,k} \mid R_1^k=t,\ \theta^k=i\} = 0$ for $t > T$.

Then since $F^k(t) = \sum_{i=0}^{k} Q_i^k(t)$,

$$EC_\alpha^{T,k} = \sum_{i=0}^{k} c_i \int_0^T e^{-\alpha t}dQ_i^k(t) + \int_0^T e^{-\alpha t}(EC_\alpha^{T-t,k})dF^k(t).$$

Multiplying by $e^{\alpha T}$ we get

$$e^{\alpha T}EC_\alpha^{T,k} = \sum_{i=0}^{k} c_i\int_0^T e^{\alpha(T-t)}dQ_i^k(t) + \int_0^T e^{\alpha(T-t)}(EC_\alpha^{T-t,k})dF^k(t)$$

Again we use Appendix A.3 and find $e^{\alpha T}EC_\alpha^{T,k}$ explicit.

$$e^{\alpha T}EC_\alpha^{T,k} = \sum_{i=0}^{k} c_i\int_0^T e^{\alpha(T-t)}dQ_i^k(t) + \int_0^T e^{\alpha(T-x)} \sum_{i=0}^{k} c_i \int_0^{T-x} e^{-\alpha t}dQ_i^k(t)dM^k(x)$$

or
$$EC_\alpha^{T,k} = \sum_{i=0}^{k} c_i\int_0^T e^{-\alpha t}dQ_i^k(t) + \sum_{i=0}^{k} c_i\int_0^T e^{-\alpha x}\int_0^{T-x} e^{-\alpha t}dQ_i^k(t)dM^k(x)$$
$$= \sum_{i=0}^{k} c_i\int_0^T e^{-\alpha t}[1+\int_0^{T-t} e^{-\alpha x}dM^k(x)]dQ_i^k(t) .$$

Alternatively we may write

$$EC_\alpha^{T,k} = E\sum_{i=0}^{k} c_i\int_0^T e^{-\alpha t}dN_i^k(t)$$

Using Appendix B.2 we see that

$$EC_\alpha^{T,k} = \sum_{i=0}^{k} c_i\int_0^T e^{-\alpha t}dM_i^k(t) .$$

Comparing this expression for $EC_\alpha^{T,k}$ and the previous when $\alpha = 0$, it follows that

$$M_i^k(T) = \int_0^T (1+M^k(T-t))dQ_i^k(t) \qquad i \leq k .$$

c) Since $Eb_\alpha^{T,k} = E\int_0^T e^{-\alpha s}R^k(Z(s))ds = -\sum_{i=k}^{M-1} \beta_i^k E\int_0^T I(Z(s)>i)e^{-\alpha s}ds$

$= -\sum_{i=k}^{M-1} \beta_i^k\int_0^T D_i^k(s)e^{-\alpha s}ds$, we can find $Eb_\alpha^{T,k}$ using that

$D_i^k(s) = \bar{F}^i(s) + \int_0^s \bar{F}^i(s-t)dM^k(t)$.

However, we will find $Eb_\alpha^{T,k}$ directly using a standard renewal argument. Conditioning on the time for the first failure (k) and using the fact that the expected cost in $[0,T]$ due to the b_i's given the first failure (k) is at time $t =$

$$E(b_\alpha^{T,k}|R_1^k=t) = \begin{cases} e^{-\alpha t}Eb_\alpha^{T-t,k} + E\{\int_0^t R^k(Y(x))e^{-\alpha x}|R_1^k=t\} & \text{if } t \le T \\ E\{\int_0^T R^k(Y(x))e^{-\alpha x}dx|R_1^k=t\} & \text{if } t > T \end{cases}$$

we get

$$\begin{aligned} Eb_\alpha^{T,k} &= \int_0^T E\{b_\alpha^{T,k}|R_1^k=t\}dF^k(t) + \int_T^\infty E\{b_\alpha^{T,k}|R_1^k=t\}dF^k(t) \\ &= \int_0^T [e^{-\alpha t}Eb_\alpha^{T-t,k} + E(\int_0^t R^k(Y(x))e^{-\alpha x}dx|R_1^k=t)]dF^k(t) \\ &\quad + \int_T^\infty E(\int_0^T R^k(Y(x))e^{-\alpha x}dx|R_1^k=t)dF^k(t) \\ &= E\int_0^{\tau^{T,k}} R^k(Y(x))e^{-\alpha x}dx + \int_0^T e^{-\alpha t}Eb_\alpha^{T-t,k}dF^k(t) \end{aligned}$$

From the proof of Theorem 3.2 we see that

$$E\int_0^{\tau^{T,k}} R^k(Y(x))e^{-\alpha x}dx = -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx .$$

Hence

$$Eb_\alpha^{T,k} = -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx + \int_0^T e^{-\alpha t}Eb_\alpha^{T-t,k}dF^k(t)$$

Multiplying this equation by $e^{\alpha T}$ and using Appendix A.3 we get

$$\begin{aligned} Eb_\alpha^{T,k} &= -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx + \int_0^T e^{-\alpha t}[-\sum_{i=k}^{M-1} \beta_i^k \int_0^{T-t} e^{-\alpha x}\bar{F}^i(x)dx]dM^k(t) \\ &= -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx - \sum_{i=k}^{M-1} \beta_i^k \int_0^T \int_0^{T-t} e^{-\alpha(t+x)}\bar{F}^i(x)dxdM^k(t) \\ &= -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}dx - \sum_{i=k}^{M-1} \beta_i^k \int_0^T e^{-\alpha x}\bar{F}^i(x) \int_0^{T-x} e^{-\alpha t}dM^k(t)dx \\ &= -\sum_{i=k}^{M-1} \beta_i^k \int_0^T \bar{F}^i(x)e^{-\alpha x}[1+\int_0^{T-x} e^{-\alpha t}dM^k(t)]dx . \end{aligned}$$

Thus we have found $w_\alpha^{T,k}$ for $\alpha \ge 0$. Setting $\alpha = 0$ we get the numerator in $\bar{B}^{T,k}$.

Corollary 5.3

If we let $S = \{M-1\}$ we get a "binary" block replacement model with M failure modes. If $b_M = 0$, we see that

$$\bar{B}^{T,M-1} = \frac{1}{T}\left[\sum_{i=0}^{M-1} c_i M_i^k(T) + c_M\right]$$

$$\bar{B}_\alpha^{T,M-1} = \frac{1}{1-e^{-\alpha T}}\left[\sum_{i=0}^{M-1} c_i \int_0^T e^{-\alpha s} dM_i^k(s) + e^{-\alpha T} c_M\right]$$

where $M_i^k(s) = \int_0^s (1+M^{M-1}(s-t))dQ_i^{M-1}(t) \qquad i \leq k$.

Remark 5.4

Since the system with probability one is in the states G_k^c, we must have

$$D_k^k(s) \equiv 1$$

$$A_k^k(T) = \int_0^T D_k^k(s)ds \equiv T \qquad \text{and}$$

$$\int_0^T e^{-\alpha s} D_k^k(s)ds \equiv \frac{1}{\alpha}[1-e^{-\alpha T}] \qquad \text{for} \quad \alpha > 0 .$$

It follows that the first integral in the expressions for $\bar{B}^{T,k}$ and $\bar{B}_\alpha^{T,k}$ equals the denominators. Hence we see that it is only the differences between the b_i's that are important for the minimizing problem. See page 10 and 11.

Furthermore we note that $\lim_{\alpha\to 0^+} \alpha\bar{B}_\alpha^{T,k} = \lim_{\alpha\to 0^+} \frac{\alpha}{1-e^{-\alpha T}} w_\alpha^{T,k} = \frac{w_0^{T,k}}{T}$ $= \bar{B}^{T,k}$ since $\frac{\alpha}{1-e^{-\alpha T}} \to \frac{1}{T}$ as $\alpha \to 0^+$ and $w_\alpha^{T,k} \to w_0^{T,k}$, (by Lebesgue Convergence Theorem).

We will now study the functions $\bar{B}^{T,k}$ and $\bar{B}_{\alpha}^{T,k}$ as functions of T. First we consider the expected average cost case.

<u>Minimizing the expected average cost per unit time, $\bar{B}^{T,k}$</u>

In the first part of this section we will state some lemmas which will help us to give conditions when a minimum is obtained for finite T or not.

<u>Lemma 5.5</u>

For $i \geq k$

$$A_i^k(T) = \int_0^T \bar{F}^i(u)(1+M^k(T-u))du$$

$$\stackrel{(i)}{=} \int_0^T \bar{F}^i(u)du + \int_0^T \int_0^{T-x} \bar{F}^i(u)du\, dM^k(x)$$

$$\stackrel{(ii)}{=} (M^k(T)+1)r_i - \gamma_i(T)$$

where $r_i = ER^i$ and $\gamma_i(T) = \int_T^\infty \bar{F}^i(u)du + \int_0^T (\int_{T-y}^\infty \bar{F}^i(u)du)dM^k(y)$.

<u>Proof</u>

(i) See the proof of Theorem 5.2 c).

(ii) $$A_i^k(T) = \int_0^T \bar{F}^i(u)du + \int_0^T \int_0^{T-x} \bar{F}^i(u)du\, dM^k(u)$$

$$= r_i - \int_T^\infty \bar{F}^i(u)du + \int_0^T (r_i - \int_{T-x}^\infty \bar{F}^i(u)du)dM^k(x)$$

$$= r_i(1+M^k(T)) - \gamma_i(T) .$$

It is possible to give an alternative proof of ii) and an interpretation of $\gamma_i(T)$.

Clearly $A_i^k(T) = E\left\{\sum_{j=1}^{N^k(T)+1} R_j^i - \gamma_T^i\right\}$

where R_j^i is the life-length in the states $\{i+1,\dots,M\}$ associated with the j-th cycle,

$$\gamma_T^i = \max\left\{0, \sum_{j=1}^{N^k(T)} R_j^k + R^i_{N^k(T)+1} - T\right\} .$$

See figure 5.1. where we have defined $S_n = \sum_{i=1}^{n} R_i^k$

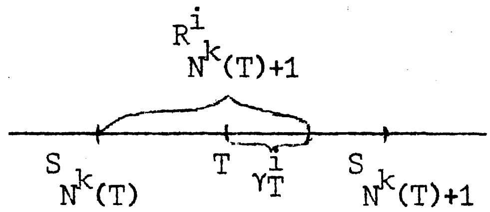

Fig. 5.1.

Using Appendix A.1, we find

$$A_i^k(T) = r_i(M^k(T)+1) - E\gamma_T^i = r_i(M^k(T)+1) - \gamma_i(T) .$$

We now assume $E(R^i)^2 < \infty$. Let $l_i = E(R^i)^2$.

<u>Lemma5.6</u>

Assume $F^k(\cdot)$ is not lattice (see Appendix A.5). Then

i) $$\lim_{T\to\infty} \left(M^k(T) - \frac{T}{r_k}\right) = \frac{l_k}{2r_k^2} - 1$$

ii) $$\lim_{T\to\infty} D_j^k(T) = \frac{r_j}{r_k} \quad \text{for } j = k,k+1,\dots,M-1$$

iii) $$\lim_{T\to\infty} \frac{A_j^k(T)}{T} = \frac{r_j}{r_k} \quad \text{for } j=k,k+1,\dots,M-1$$

iv) $$\lim_{T\to\infty} \left(A_j^k(T) - T\frac{r_j}{r_k}\right) = r_j \frac{l_k}{2r_k^2} - \frac{1}{2}\frac{l_j}{r_k} \qquad j=k,k+1,\dots,M-1$$

v) $$\lim_{T\to\infty} \frac{EC_0^{T,k}}{T} = \frac{1}{r_k}\sum_{i=0}^{k} c_i P_i^k = \frac{c_k^*}{r_k} \quad \left(\text{or } \lim_{T\to\infty}\frac{M_i^k(T)}{T} = \frac{P_i^k}{r_k}\right)$$

where $EC_0^{T,k}$ = expected cost in $[0,T]$ due to the c_i's for $i \leq k$

$$= \sum_{i=0}^{k} c_i M_i^k(T)$$

vi) $$\lim_{T\to\infty}\left(EC_0^{T,k} - \frac{c_k^* T}{r_k}\right) = c_k^* \frac{l_k}{2r_k^2} - \frac{\sum_{i=0}^{k} c_i h_i^k}{r_k}$$

$$\left(\text{or } \lim_{T\to\infty}\left(M_i^k(T) - \frac{P_i^k T}{r_k}\right) = P_i^k \frac{l_k}{2r_k^2} - \frac{h_i^k}{r_k} = P_i^k\left[\frac{l_k}{2r_k^2} - \frac{E(R^k \mid \theta^k = i)}{r_k}\right]\right)$$

Proof

i) See Karlin (1975) page 195.

ii) Since $D_j^k(T) = \bar{F}^j(T) + \int_0^T \bar{F}^j(T-x)dM^k(x)$

and $\bar{F}^j(T)$ is directly Riemann integrable (see Appendix A.5a)), it follows by Appendix A.5 that

$$D_j^k(T) \xrightarrow[T\to\infty]{} \frac{1}{r_k}\int_0^\infty \bar{F}^j(u)du = \frac{r_j}{r_k}.$$

iii) If we consider the time in states > j as cost (reward), then by Appendix A.2

$$\frac{A_j^k(T)}{T} = \frac{E\left[\sum_{n=1}^{N^k(T)} R_n^j + \delta^j_{N^k(T)}\right]}{T} \xrightarrow[T\to\infty]{} \frac{ER^j}{ER^k}$$

where $\delta^j_{N^k(T)}$ is the time in states > j in the interval $\left(\sum_{n=1}^{N^k(T)} R_n^k, T\right]$.

iv) Let $g_j(T) = \int_T^\infty \bar{F}^j(u)du$, then according to Lemma 5.5 we may write

$$A_j^k(T) = (1+M^k(T)r_j - \gamma_j(T) \quad \text{where}$$

$$\gamma_j(T) = g_j(T) + \int_0^T g_j(T-y)dM^k(u) .$$

Since $\int_0^\infty g_j(u)du = \int_0^\infty \int_u^\infty \bar{F}^j(x)dx\,du = \int_0^\infty x\,\bar{F}^j(x)dx$

$$= \int_0^\infty x \int_0^\infty I(u,x : u>x)dF^j(u)dx = \int_0^\infty \int_0^u x\,dx\,dF^j(u) = \int_0^\infty \tfrac{1}{2}u^2\,dF^j(u)$$

$= \frac{1}{2}l_j < \infty$, $g_j(\cdot)$ is decreasing and $g_j(\cdot) \geq 0$, we see that $g_j(\cdot)$ is directly Riemann integrable by Appendix A.5 a).

Hence $\gamma_j(T) \xrightarrow[T\to\infty]{} \frac{1}{r_k}\int_0^\infty g_j(u)du = \frac{l_j}{2r_k}$ by the Key Renewal Theorem, Appendix A.5.

Using i) it follows that

$$A_j^k(T) - \frac{r_j}{r_k}T = (M^k(T) - \frac{T}{r_k})r_j + r_j - \gamma_j(T) \xrightarrow[T\to\infty]{} r_j\frac{l_k}{2r_k^2} - \frac{1}{2}\frac{l_j}{r_k} .$$

v) We will use Appendix A.2.

Let L_n = cost associated with the n-th replacement due to c_i $i \leq k$.

Then $\frac{1}{T}EC_0^{T,k} = \frac{1}{T}\sum_{n=1}^{N^k(T)} L_n \xrightarrow[T\to\infty]{} \frac{EL_1}{r_k} = \frac{E\sum_{i=0}^k c_i I(\theta_k = i)}{r_k} = \frac{\sum_{i=0}^k c_i P_i^k}{r_k}$

It follows that $\lim_{T\to\infty} \frac{M_i^k(T)}{T} = \frac{P_i^k}{r_k}$.

vi) Let $\bar{\bar{Q}}_i^k(t) = P_i^k - Q_i^k(t) = P_i^k\,\bar{\bar{F}}_i^k(t)$. Then

$$\begin{aligned}
EC_0^{T,k} &= \sum_{i=0}^{k} c_i[Q_i^k(T) + \int_0^T M^k(T-u)dQ_i^k(u)] \\
&= \sum_{i=0}^{k} c_i Q_i^k(T) + \int_0^T Q_i^k(T-u)dM^k(u) \\
&= \sum_{i=0}^{k} c_i P_i^k + \sum_{i=0}^{k} c_i(Q_i^k(T)-P_i^k) - \sum_{i=0}^{k} c_i\int_0^T \bar{\bar{Q}}_i^k(T-x)dM^k(x) + \sum_{i=0}^{k} c_i P_i^k M^k(T) \\
&= \sum_{i=0}^{k} c_i P_i^k(1+M^k(T)) - \sum_{i=0}^{k} c_i\int_0^T \bar{\bar{Q}}_i^k(T-x)dM^k(x) - \sum_{i=0}^{k} c_i \bar{\bar{Q}}_i^k(t)
\end{aligned}$$

Since $\int_0^\infty \bar{\bar{Q}}_i^k(u)du = P_i^k\int_0^\infty \bar{\bar{F}}_i^k(u)du = P_i^k\int_0^\infty udF_i^k(u) = \int_0^\infty udQ_i^k(u)$

$= h_i^k < \infty$ and $\bar{\bar{Q}}_i^k(u) \geq 0$ and decreasing, it follows that $\bar{\bar{Q}}_i^k(u)$ is directly Riemann integrable. Using Appendix A.5 and i) we get

$$\lim_{T\to\infty} (EC_0^{T,k} - \frac{c_k^* T}{r_k})$$

$$= \lim_{T\to\infty}\left[c_k^*(M^k(T) - \frac{T}{r_k}) + c_k^* - \frac{\sum_{i=0}^{k} c_i h_i^k}{r_k} + 0_T(1)\right]^{1)}$$

$$= c_k^*(\frac{l_k}{2r_k^2} - 1) + c_k^* - \frac{\sum_{i=0}^{k} c_i h_i^k}{r_k}$$

Note that these results can be modified if $F^k(\cdot)$ is lattice using the analogous version of Lemma 5.6 i), see Feller (1968) (chapter XIII (12.2)).

<u>Lemma 5.7</u>

i) $\lim_{T\to\infty} \bar{B}^{T,k} = \frac{1}{r_k}[-\sum_{i=k}^{M-1} \beta_i^k r_i + c_k^*] = B^{\infty,k} = \bar{B}^{\infty,k}$ where $c_k^* = \sum_{i=0}^{k} c_i P_i^k$.

ii) $\lim_{T\to 0^+} \bar{B}^{T,k} = \infty$.

1) By $0_T(1)$ we mean a function which converges to 0 as $T\to\infty$.

Proof

i) From Theorem 5.2 we have

$$\bar{B}^{T,k} = \frac{1}{T}\left[-\sum_{i=k}^{M-1} \beta_i^k A_i^k(T) + EC_0^{T,k} - \sum_{i=k}^{M-1} \gamma_i^k D_i^k(T)\right].$$

i) follows now from Lemma 5.6 ii), iii) and v).

We can also prove i) using Appendix A.2 directly. To see this let $V_i^{\infty,k}$ be the cost of the i-th cycle of length R_i^k, and $\delta_{N^k(T)}$ the cost in $(\sum_{i=1}^{N^k(T)} R_i^k, T]$. Then

$$\bar{B}^{T,k} = \frac{1}{T} E\left\{\sum_{i=1}^{N^k(T)} V_i^{\infty,k} + \delta_{N^k(T)}\right\} \to \frac{EV^{\infty,k}}{r_k} = B^{\infty,k}$$

by Appendix A.2 and Theorem 3.2 (Corollary 3.3).

ii) Obvious.

We now see from the expression for $B^{T,k}$ in Theorem 5.2 that if $Q_i^j(\cdot)$ $\forall j,i$ is continuous, then $\bar{B}^{T,k}$ is continuous and hence $\bar{B}^{T,k}$ must have a minimum (included $T=\infty$). Note that $M^k(\cdot)$ is continuous if $F^k(\cdot)$ is continuous. See Appendix A.6.

Let

$$d = \frac{l_k}{2r_k^2} \sum_{i=k}^{M-1} \beta_i^k r_i + \frac{1}{2r_k} \sum_{i=k}^{M-1} \beta_i^k l_i - \frac{1}{r_k} \sum_{i=k}^{M-1} \gamma_i^k r_i + \frac{l_k}{2r_k^2} \sum_{i=0}^{k} c_i P_i^k - \frac{1}{r_k} \sum_{i=0}^{k} c_i h_i^k$$

Lemma 5.8

If $d > 0$ then $\bar{B}^{T,k}$ converges to $B^{\infty,k}$ from above.

If $d < 0$ then $\bar{B}^{T,k}$ converges to $B^{\infty,k}$ from below.

Proof

We use Lemma 5.6 ii), iv) and vi) and Lemma 5.7.

From Lemma 5.6 ii), iv) and vi)

$$D_j^k(T) = \frac{r_j}{r_k} + 0_T(1) , \quad j \geq k$$

$$A_j^k(T) = T\frac{r_j}{r_k} + r_j\frac{l_k}{2r_k^2} - \tfrac{1}{2}\frac{l_j}{r_k} + 0_T(1) , \quad j \geq k$$

$$EC_0^{T,k} = \frac{c_k^*}{r_k}T + \frac{c_k^* l_k}{2r_k^2} - \frac{1}{r_k}\sum_{i=0}^{k} c_i h_i^k + 0_T(1) \quad \text{where} \quad c_k^* = \sum_{i=0}^{k} c_i P_i^k$$

Inserting these expressions into $\bar{B}^{T,k}$ and remembering Lemma 5.7 we get

$$\begin{aligned} \bar{B}^{T,k} &= \frac{1}{T}\left[-\sum_{i=k}^{M-1} \beta_i^k A_i^k(T) + EC_0^{T,k} - \sum_{i=k}^{M-1} \gamma_i^k D_i^k(T)\right] \\ &= -\frac{1}{r_k}\sum_{i=k}^{M-1} \beta_i^k r_i + \frac{c_k^*}{r_k} + \frac{d}{T} + \frac{0_T(1)}{T} \\ &= B^{\infty,k} + \frac{d}{T} + \frac{0_T(1)}{T} . \end{aligned}$$

Assume $d > 0$. Then there exists a T_0 such that

$0_T(1) < d$ for $T > T_0$. Clearly $\bar{B}^{T,k} > B^{\infty,k}$ for $T > T_0$.

If $d < 0$, then there exists a T_0' such that

$0_T(1) < -d$ for $T > T_0'$. Here $\bar{B}^{T,k} < B^{\infty,k}$ for $T > T_0'$.

Since $\lim_{T\to\infty} \bar{B}^{T,k} = B^{\infty,k}$, the lemma is proved.

We now assume absolute continuous distributions.

Let $m^k(x) = \frac{d}{dx} M^k(x)$ (see Appendix A.6) and

$$\bar{a}^k(x) = \frac{d}{dx} w_0^{x,k} = -\sum_{i=k}^{M-1} \beta_i^k D_i^k(x) + \sum_{i=0}^{k} c_i m_i^k(x) - \sum_{i=k}^{M-1} \gamma_i^k d_i^k(x)$$

where $d_i^k(x) = \frac{d}{dx} D_i^k(x) = m^k(x) - (f^i(x) + \int_0^x f^i(x-y)dM^k(y)), i > k, \; d_k^k(x) \equiv 0$

$$m_i^k(x) = \frac{d}{dx} M_i^k(x) = q_i^k(x) + \int_0^x m^k(x-u)dQ_i^k(u)$$

$$= q_i^k(x) + \int_0^x q_i^k(x-u)dM^k(u), \quad i \leq k .$$

Note that $m_0^0(x) = \frac{d}{dx} M_0^0(x) = \frac{d}{dx} M^0(x) = m^0(x)$.

We will assume that $\lim_{x\to\infty} \bar{a}^k(x)$ exists. In the following lemma we find this limit.

Lemma 5.9

Assume all probability densities defined are directly Riemann integrable and tending to 0 at infinity. Then

$$\lim_{x\to\infty} \bar{a}^k(x) = B^{\infty,k} = \bar{B}^{\infty,k} .$$

Proof

By Lemma 5.6 ii), $D_i^k(x) \to \frac{r_i}{r_k}$ as $x \to \infty$.

We can conclude that $d_i^k(x) \to 0$ as $x \to \infty$ by Appendix A.5 and A.6 since $\int_0^x f^i(x-y)dM^k(y) \xrightarrow[x\to\infty]{} \frac{1}{r_k}$, $m^k(x) \xrightarrow[x\to\infty]{} \frac{1}{r_k}$ and $f^i(x) \xrightarrow[x\to\infty]{} 0$.

Furthermore $m_i^k(x) \xrightarrow[x\to\infty]{} \frac{\int_0^\infty q_i^k(u)du}{r_k} = \frac{P_i^k}{r_k}$ by Appendix A.5.

Thus $\lim_{x\to\infty} \bar{a}^k(x) = -\frac{1}{r_k} \sum_{i=k}^{M-1} \beta_i^k r_i + \frac{c_k^*}{r_k} = B^{\infty,k}$.

In the following we will assume all probability densities defined are directly Riemann integrable and tending to 0 at infinity.

Theorem 5.10

a) Assume the distributions $Q_i^j(\cdot)$ $\forall i,j$ are continuous. Then if $d<0$, we have a finite minimum of $B^{\cdot,k}$.

Assume absolutely continuous distributions $Q_i^j(\cdot)$.

b) If the equation $\bar{a}^k(x) = \bar{B}^{x,k}$ has no finite solution, then $x=\infty$ minimizes $\bar{B}^{\cdot,k}$.

c) If $d<0$, and $\bar{a}^k(x)$ is strictly increasing in x, then we have a unique, finite minimum.

d) If $d>0$, and $\bar{a}^k(x)$ is increasing, then $x=\infty$ minimizes $B^{\cdot,k}$, and $\bar{B}^{x,k}$ is strictly decreasing in x.

Proof

a) If $d<0$, then by Lemma 5.8 $\bar{B}^{x,k}$ converges to $B^{\infty,k}$ from below, so there must be a finite minimum since $\bar{B}^{x,k}$ is continuous.

We now assume absolutely continuous distributions $Q_i^j(\cdot)$. Then we must have $\frac{d}{dx}\bar{B}^{x,k} = 0$ at a minimum.

A straightforward calculation gives

$$\frac{d}{dx}\bar{B}^{x,k} = 0 \iff \bar{B}^{x,k} = \bar{a}^k(x)$$

$$\frac{d}{dx}\bar{B}^{x,k} > 0 \iff \bar{B}^{x,k} < \bar{a}^k(x)$$

$$\frac{d}{dx}\bar{B}^{x,k} < 0 \iff \bar{B}^{x,k} > \bar{a}^k(x)$$

Hence $\bar{a}^k(x)$ and $\bar{B}^{x,k}$ intersect at all extremum points of $\bar{B}^{x,k}$ so that $\bar{a}^k(x)$ crosses from below (above) at the minima (maxima).

Using this fact the proof of b), c) and d) is similar to the proof of Theorem 3.7.

b) Follows from the above relations and Lemma 5.7.

c) See the proof of Theorem 3.7 iii).

d) If $d > 0$, then by Lemma 5.8 $\bar{B}^{x,k}$ converges to $B^{\infty,k}$ from above. Since $\bar{a}^k(x)$ is increasing we must have $\bar{a}^k(x) < \bar{B}^k(x)$ $\forall x$. Hence $\frac{d}{dx}\bar{B}^{x,k} < 0$, and $\bar{B}^{x,k}$ is strictly increasing in x and $x = \infty$ minimizes $\bar{B}^{\cdot,k}$. See Fig. 5.3.

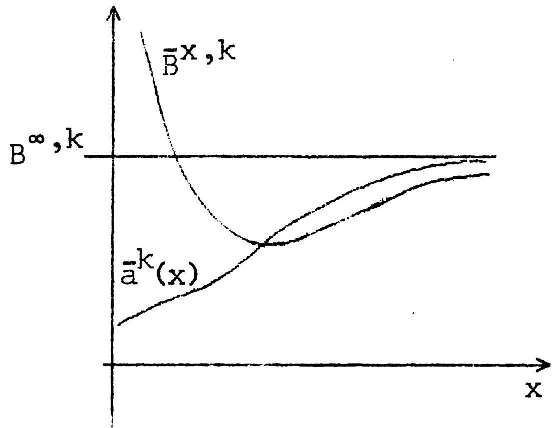

Fig. 5.2 Illustration of Theorem 5.10 c)

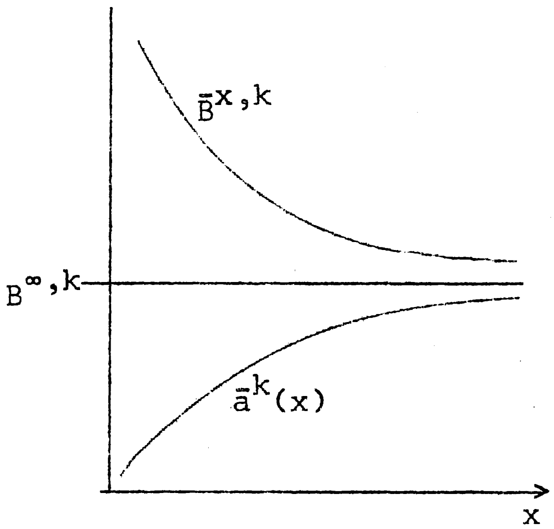

Fig. 5.3 Illustration of Theorem 5.10 d)

Remark 5.11

For the model $S = \{M-1\}$ we find (if $b_M = 0$)

$$\bar{a}^{M-1}(t) = \sum_{i=0}^{M-1} c_i m_i^k(t)$$

$$d = c_M + \frac{l_{M-1}}{2r_{M-1}^2} \sum_{i=0}^{M-1} c_i P_i^{M-1} - \frac{1}{r_{M-1}} \sum_{i=0}^{M-1} c_i h_i^{M-1}$$

$$B^{\infty,k} = \frac{1}{r_{M-1}} \sum_{i=0}^{M-1} c_i P_i^{M-1}$$

Marginal cost analysis for MBRP

We define the marginal cost of a block replacement at x as $\lim_{\Delta \to 0^+} \frac{\bar{G}(\Delta)}{\Delta}$ where

$\bar{G}(\Delta)$ = Expected cost associated with waiting a short time Δ
- Expected cost of a block replacement at x.

Theorem 5.12

Assume absolutely continuous distributions $Q_i^j(\cdot)$. Then the marginal cost of a block replacement at x is $\bar{a}^k(x)$.

Proof

Since the expected cost in $[0,x)$ when the block interval is $x+\Delta$ equals the expected cost in $[0,x)$ when the block interval is x,

$\bar{G}(\Delta) = w_0^{x+\Delta,k} - w_0^{x,k}$ where $w_0^{x,k}$ = expected cost in a block interval of length x.

Hence $\frac{\bar{G}(\Delta)}{\Delta} \to \frac{d}{dx} w_0^{x,k} = \bar{a}^k(x)$.

We now turn to the discounted case. We will just list the main definitions and results.

Minimizing the total expected discounted cost function $\bar{B}_\alpha^{x,k}$

Lemma 5.13

$$\lim_{x\to\infty} \bar{B}_\alpha^{x,k} = \bar{B}_\alpha^{\infty,k} = B_\alpha^{\infty,k} .$$

Proof

Since $1-e^{-\alpha x} \to 1$ as $x \to \infty$, we see from Lemma 5.1 that we must show that $w_\alpha^{x,k} \to B_\alpha^{\infty,k}$ as $x \to \infty$.

Let $V_{\alpha,i}^{\infty,k}$ be the discounted cost of the i-th cycle of length R_i^k. Then

$$E\left\{\sum_{n=1}^{N^k(x)} e^{-\alpha[R_1^k+\ldots+R_{n-1}^k]} V_{\alpha,n}^{\infty,k}\right\} \leq w_\alpha^{x,k} \leq E\left\{\sum_{n=1}^{N^k(x)+1} e^{-\alpha[R_1^k+\ldots+R_{n-1}^k]} V_{\alpha,n}^{\infty,k}\right\}$$

where $R_0^k \stackrel{def}{=} 0$.

Now $N^k(x) \xrightarrow[x\to\infty]{} \infty$ with probability one (Laha and Rohatgi (1979) page 119). Hence using Monotone Convergence Theorem twice, we get for $\delta = 0,1$

$$\lim_{x\to\infty} E\left\{\sum_{n=1}^{N^k(x)+\delta} e^{-\alpha[R_1^k+\ldots+R_{n-1}^k]} V_{\alpha,n}^{\infty,k}\right\} = E\left\{\sum_{n=1}^{\infty} e^{-\alpha[R_1^k+\ldots+R_{n-1}^k]} V_{\alpha,n}^{\infty,k}\right\}$$

$$= \sum_{n=1}^{\infty}\left[E\, e^{-\alpha R^k}\right]^{n-1} EV_\alpha^{\infty,k} = \frac{EV_\alpha^{\infty,k}}{1-Ee^{-\alpha R^k}} = B_\alpha^{\infty,k}$$

Thus $w_\alpha^{x,k} \to B_\alpha^{\infty,k}$ as $x \to \infty$.

We see from the expression for $\bar{B}_\alpha^{x,k}$ in Theorem 5.2 that if $Q_i^j(\cdot)$ $\forall i,j$ is continuous, then $\bar{B}^{x,k}$ is continuous. Thus $\bar{B}^{x,k}$ must have a minimum (included $x=\infty$).

We now assume absolutely continuous distributions $Q_i^j(\cdot)$.

Define $\bar{a}_\alpha^k(\cdot)$ by

$$\bar{a}_\alpha^k(x) = e^{\alpha x}\frac{d}{dx}w_\alpha^{x,k} = -\sum_{i=k}^{M-1}\beta_i^k D_i^k(x) + \sum_{i=0}^{k} c_i m_i^k(x) - \sum_{i=k}^{M-1}\gamma_i^k d_i^k(x) + \alpha\sum_{i=k}^{M-1}\gamma_i^k D_i^k(x)$$

$$= \bar{a}^k(x) + \alpha\sum_{i=k}^{M-1}\gamma_i^k D_i^k(x)$$

where $m_i^k(x) = \frac{d}{dx} M_i^k(x)$, $d_i^k(x) = \frac{d}{dx} D_i^k(x)$.

Clearly $\lim\limits_{\alpha\to 0^+} \bar{a}_\alpha^k(x) = \bar{a}^k(x)$.

Lemma 5.14

$$\lim_{x\to\infty} \bar{a}_\alpha^k(x) = B^{\infty,k} + \alpha\sum_{i=k}^{M-1}\gamma_i^k \frac{r_i}{r_k} .$$

Proof

Use Lemma 5.6 ii) and Lemma 5.9 .

Theorem 5.15

Write Theorem 3.11 with $\bar{B}_\alpha^{x,k}$ and $\bar{a}_\alpha^{x,k}$ in stead of $B_\alpha^{x,k}$ and $a_\alpha^k(x)$.

Theorem 5.16

The marginal cost of a block replacement at x in the discounted case equals $\bar{a}_\alpha^k(x)$.

Proof

Let $\bar{G}_\alpha(\cdot)$ = Expected discounted associated with waiting a short time Δ - Expected discounted cost of a block replacement at x . Then

$$\lim_{\Delta\to 0^+}\frac{G_\alpha(\cdot)}{\Delta} = \lim_{\Delta\to 0^+}\{e^{\alpha x}[w_\alpha^{x+\Delta,k} - w_\alpha^{x,k}]\frac{1}{\Delta}\} = \bar{a}_\alpha^k(x) .$$

See the proof of Theorem 5.12 for the precise details. Note that we use time x as starting point for the discounting.

6. A simple periodic replacement model

If the state of the system is known only at inspection (it may be too expensive to monitor the system continuously), the following replacement rule may be of interest.

"Replace at nT , $n = 1,2,\ldots$ "

"$T = \infty$" means "never replace".
Note that we in this policy have no control limit.

It follows that the expected long run average cost $B^{T,-1}$ and the total expected discounted cost, $B_\alpha^{T,-1}$ are given by

$$B^{T,-1} = \frac{1}{T}\left\{-\sum_{i=-1}^{M-1} \beta_i^{-1}\int_0^T \bar{F}^i(x)dx + \sum_{i=-1}^{M} \gamma_i^{-1} F^i(T) + c_M\right\}$$

$$B_\alpha^{T,-1} = \frac{1}{1-e^{-\alpha T}}\left\{-\sum_{i=-1}^{M-1} \beta_i^{-1}\int_0^T e^{-\alpha x}\bar{F}^i(x)dx + e^{-\alpha T}\left[\sum_{i=-1}^{M-1} \gamma_i^{-1} F^i(T) + c_M\right]\right\}$$

for $T < \infty$

where we have $\beta_{-1}^{-1} = -b_0$, $\beta_i^{-1} = b_i - b_{i+1}$, $i = 0,1,\ldots,M-1 \Rightarrow b_i = -\sum_{j=-1}^{i-1} \beta_j^{-1}$

$\gamma_{-1}^{-1} = -c_0$, $\gamma_i^{-1} = c_i - c_{i+1}$, $i = 0,1,\ldots,M-1 \Rightarrow c_i = -\sum_{j=-1}^{i-1} \gamma_j^{-1}$

and $\bar{F}^{-1}(\cdot) \equiv 1$ $(P\{R^{-1} = \infty\} = 1)$

To see this let $w_\alpha^{T,-1}$ be the expected discounted cost in $[0,T]$ under this policy.

Then $B^{T,-1} = \frac{w_0^{T,-1}}{T}$ and $B_\alpha^{T,-1} = \frac{w_\alpha^{T,-1}}{1-e^{-\alpha T}}$ as in Lemma 5.1.

Clearly

$$w_\alpha^{T,-1} = E\int_0^T R^{-1}(Y(x))e^{-\alpha x}dx + E\sum_{i=0}^{M} c_i I(Y(T)=i)e^{-\alpha T}$$

$$= -\sum_{i=-1}^{M-1} \beta_i^{-1}\int_0^T \bar{F}^i(x)e^{-\alpha x}dx - e^{-\alpha T}\left(\sum_{i=-1}^{M-1} \gamma_i^{-1}\bar{F}^i(T)\right)$$

$$= -\sum_{i=-1}^{M-1} \beta_i^{-1}\int_0^T \bar{F}^i(x)e^{-\alpha x}dx + e^{-\alpha T}\left(\sum_{i=-1}^{M-1} \gamma_i^{-1} F^i(T) + c_M\right).$$

Note that we could have found $w_\alpha^{T,-1}$ directly from the numerators in (3.3) and (3.4) by letting $k = -1$ and $\sum_{i=0}^{-1} \cdot \equiv 0$.

It is intuitively obvious that the optimal T should be finite; in Theorem 6.1 (6.2) we show that this holds under reasonable conditions.

First we give some simple results.

Since $F^{-1}(\cdot) \equiv 0$, we see that

$$\lim_{T\to\infty} B^{T,-1} = B^{\infty,-1} = \frac{b_0}{\alpha} - \sum_{i=0}^{M} \beta_i^{-1} \int_0^\infty e^{-\alpha x} \bar{F}^i(x)dx$$

Obviously $\lim\limits_{T\to 0^+} B^{T,-1} = \infty$ and $\lim\limits_{T\to 0^+} B_\alpha^{T,-1} = \infty$.

If $F^i(\cdot)$ $\forall i$ is continuous, then $B^{T,-1}$ $(B_\alpha^{T,-1})$ is continuous. Hence $B^{T,-1}$ $(B_\alpha^{T,-1})$ has a minimum (included $T = \infty$). If $F^i(\cdot)$ $\forall i$ is absolutely continuous $(f^{-1} \equiv 0)$, then let

$$a^{-1}(x) = -\sum_{i=-1}^{M-1} \beta_i^{-1} \bar{F}^i(x) + \sum_{i=-1}^{M-1} \gamma_i^{-1} f^i(x) .$$

Assume $f^j(x) \to 0$ as $x \to \infty$. Then using the fact that $\bar{F}^{-1}(x) \equiv 1$, we see that $\lim\limits_{x\to\infty} a^{-1}(x) = b_0$.

For the discounted case we define

$$a_\alpha^{-1}(x) = -\sum_{i=-1}^{M-1} \beta_i^{-1} \bar{F}^i(x) + \sum_{i=-1}^{M-1} \gamma_i^{-1} f^i(x) - \alpha\left[\sum_{i=-1}^{M-1} \gamma_i^{-1} F^i(x) + c_M\right]$$

Since $F^{-1}(x) \equiv 0$ and $\sum_{i=0}^{M-1} \gamma_i^{-1} = -c_M - \gamma_{-1}^{-1} = -c_M + c_0$,

we see that $\lim\limits_{x\to\infty} a_\alpha^{-1}(x) = b_0 - \alpha c_0$.

We state the main theorem for this model. The proof is standard. First we consider the expected average cost case.

Theorem 6.1

Let $d = -\sum_{i=0}^{M-1} \beta_i^{-1} r_i + c_0$.

a) Assume $F^i(\cdot)$ $\forall i$ is continuous.

If $d < 0$, we have a finite minimum of $B^{\cdot,-1}$.

Assume in the following that $F^i(\cdot)$ $\forall i$ is absolutely continuous.

b) If the equation $a^{-1}(x) = B^{x,-1}$ has no finite solution, then $x = \infty$ minimizes $B^{\cdot,-1}$.

c) If $d < 0$ and $a^{-1}(x)$ is strictly increasing, then we have a unique, finite minimum.

d) If $d > 0$ and $a^{-1}(x)$ is increasing, then $x = \infty$ minimizes $B^{\cdot,-1}$, and $\bar{B}^{x,-1}$ is strictly decreasing in x .

Proof

a) Since $\int_0^x \bar{F}^i(x)dx \to r_i$ as $x \to \infty$ and $\sum_{i=-1}^{M-1} \gamma_i^{-1} F^i(x) + c_M \to c_0$ as $x \to \infty$, we may write

$$B^{x,-1} = b_0 + \frac{-\sum_{i=0}^{M-1} \beta_i^{-1} r_i + c_0}{x} + \frac{0_x(1)}{x}$$

It follows that if $d < 0$, then $B^{x,-1}$ converges to $b_0 = B^{\infty,-1}$ from below so we must have a finite minimum.

b), c) and d). Using the fact that $B^{x,-1}$ and $a^{-1}(x)$ intersect at all extremum points of $B^{x,-1}$ so that $a^{-1}(x)$ crosses from below (above) at the minima (maxima), the proof is standard, see for example the proof of Theorem 5.10 b), c) and d).

We now turn to the discounted case.

Theorem 6.2

Write Theorem 3.11 with $k = -1$.

Note that the condition $\lim_{x\to\infty} a_\alpha^{-1}(x) > \alpha B_\alpha^{\infty,-1}$ is equivalent to

$$c_0 < \sum_{i=0}^{M-1} \beta_i^{-1} \int_0^\infty e^{-\alpha x} \bar{F}^i(x)dx .$$

7. Some operating characteristics of MARP and MBRP

Since we for each control limit k in principle have a binary situation, some interesting characteristics of MARP and MBRP follow from the binary theory. See Barlow and Proschan (1975), (1965).

The following concepts will be helpful:

Given random variables S and U , then write

$S \overset{st}{\leq} U \iff P\{S\geq n\} \leq P\{U\geq n\}$ for each $n > 0$.

We say S is stochastically less than U .

$S \uparrow st$ in $U \iff P\{S>s \mid U=u\}$ is increasing in u $\forall s$.

We say S is stochastically increasing in U .

$N^k(t)$ = the number of failures (k) in $[0,t]$ when we replace at failures (k) only.

$N_A^{T,k}(t)$ = the number of failures (k) in $[0,t]$ under MARP

$N_B^{T,k}(t)$ = " " " " " " " MBRP

$R_A^{T,k}(t)$ $(R_B^{T,k}(t))$ = the number of removals during $[0,t]$ under MARP (MBRP) (including both failures (k) and preventive replacements).

Then from Barlow and Proschan (1975) chapter 6

7.1 $N^k(t) \overset{st}{\geq} N_A^{T,k}(t) \quad \forall t \geq 0,\ T \geq 0 \Leftrightarrow F^k$ is NBU.

(See Barlow and Proschan (1975) page 159 for the definition of an NBU distribution.)

7.1 states that the class of NBU distributions is the largest class for which age replacement dimenishes stochastically the number of failures(k) experienced in any particular time interval $[0,t]$, $0 < t < \infty$.

Next we state that a similar characterization applies in the case of block replacement.

7.2 $N^k(t) \overset{st}{\geq} N_B^{T,k}(t)$ for all $t \geq 0,\ T \geq 0 \Leftrightarrow F^k$ is NBU.

7.3 $N_B^{nT,k}(t) \overset{st}{\geq} N_B^{T,k}(t)$ for $t \geq 0,\ T \geq 0 \quad n = 1,2,\ldots \Leftrightarrow F^k$ is NBU.

Next we study the effect of varying the replacement interval T under MARP.

7.4 $N_A^{T,k}(t) \uparrow st$ in $T \geq 0$ for each $t \geq 0 \Leftrightarrow F^k$ is IFR.

Thus under an MARP with an IFR failure (k) distribution, the number of failures (k) observed at any time interval $[0,t]$ increase stochastically as the replacement interval T increases.

A weaker comparison is possible for the NBU class:

7.5 $N_A^{nT,k}(t) \overset{st}{\geq} N_A^{T,k}(t)$ for $t \geq 0,\ T \geq 0,\ n = 1,2,\ldots \Leftrightarrow F^k$ is NBU.

Comparisons between age and block replacement policies are made in Barlow and Proschan (1965).

7.6 $R_A^{T,k}(t) \overset{st}{\leq} R_B^{T,k}(t) \quad \forall t,\ T > 0$.

7.7 $N_A^{T,k}(t) \overset{st}{\geq} N_B^{T,k}(t)$ if F^k is IFR.

Thus for every underlying life distribution, block replacement leads to more removals (stochastically) than does age replacement.

If the life distribution is actually IFR, then block replacement reduces (stochastically) the number of failures (k) experienced.

Now let $N_{A,i}^{T,k}(t)$ = the number of failures(i) under MARP with control limit k , $M-1 \geq i \geq k$. Then we have (remember 7.4)

7.8 $N_{A,i}^{T,k}(t) \uparrow st$ in T for each $t \geq 0 \Leftrightarrow F^i$ is IFR.

Proof

Let $S_i^{T,k}(t)$ be the probability that the system fails (i) before t or at t under MARP, then

$$S_i^{T,k}(t) = 1-(\bar{F}^i(T))^n \bar{F}^i(t-nT) \quad \text{for} \quad nT \leq t < (n+1)T .$$

The proof now goes exactly as in the proof of Theorem 4.6 page 181 in Barlow and Proschan (1975).

Note that we from the probability $S_i^{T,k}(t)$ can find the expected time to the first failure (i). Clearly this expected time equals $\int_0^\infty (1-S_i^{T,k}(t)dt$

$$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} (\bar{F}^i(T)^n \bar{F}^i(t-nT)dt = \sum_{n=0}^{\infty} \bar{F}^i(T) \int_{nT}^{(n+1)T} \bar{F}^i(t-nT)dt$$

$$= \sum_{n=0}^{\infty} (\bar{F}^i(T))^n \int_0^T \bar{F}^i(t)dt = \frac{\int_0^T \bar{F}^i(t)dt}{F^i(T)}$$

See Barlow and Proschan (1965) page 61-62.

We now state some special characteristics for MBRP. First we give some results from Theorem 5.2.

(i) The expected time the system is in the states $> i$ $(i \geq k)$ in $[0,t]$ when we replace at failure (k) only =

$$A_i^k(t) = \int_0^t \bar{F}^i(x)(1+M^k(t-x))dx$$

(ii) The probability that the system is in state $> i$ $(i \geq k)$ at t when we replace at failure (k) only =

$$D_i^k(t) = \bar{F}^i(t) + \bar{F}^i * M^k(t)$$

(iii) The expected number of replacements from state i $(i \leq k)$ in $[0,t]$ when we replace at failure (k) only =

$$M_i^k(t) = \int_0^t (1+M^k(t-x))dQ_i^k(u)$$

Under MBRP the replacement policy is of the form "Replace at failure (k) only" in the block intervals and perform preventive replacements at nT, $n = 1,2,\ldots$.
Hence for $nT \leq t < (n+1)T$ we must have

(i) The expected time the system is in the states $> i$ $(i \geq k)$ in $[0,t]$ under MBRP = $n\, A_i^k(T) + A_i^k(t-nT)$.

(ii) The expected number of replacements from states $> i$ $(i \geq k)$ in $[0,t]$ under MBRP = $n\, D_i^k(T)$.

(iii) The expected number of replacements from state i $(i \leq k)$ in $[0,t]$ under MBRP = $n\, M_i^k(T) + M_i^k(t-nT)$.

8. Time dependent cost functions for MARP and MBRP

We here state the expected average long run cost per unit time (the total expected, discounted cost) for MARP and MBRP when the costs are time dependent.

If the system is replaced by a new one at system age x when it is in state i, the cost equals $c_i(x)$. When the system is in state i at x, there is a cost $b_i(x)$.

For $k = 0,1,\dots,M-1$ let

$$C^k(Y(x)) = \sum_{i=0}^{M} c_i(x)I(Y(x)=i)$$
$$= \sum_{i=0}^{k} c_i(x)I(Y(x)=i) - \sum_{i=k}^{M-1} \gamma_i^k(x)I(R^i>x)$$

$$R^k(Y(x)) = \sum_{i=k+1}^{M} b_i(x)I(Y(x)=i) = - \sum_{i=k}^{M-1} \beta_i^k(x)I(R^i>x)$$

where of course

$$\gamma_k^k(x) = -c_{k+1}(x) \quad \text{and} \quad \gamma_i^k(x) = c_i(x) - c_{i+1}(x) \quad \text{for } i=k+1,\dots,M-1$$
$$\beta_k^k(x) = -b_{k+1}(x) \quad \text{and} \quad \beta_i^k(x) = b_i(x) - b_{i+1}(x) \quad \text{for } i=k+1,\dots,M-1$$

$$b_M(x) \le b_{M-1}(x) \le \dots \le b_0(x) \;, \quad 0 < c_M(x) \le \dots \le c_0(x) \quad \forall x \,.$$

First we consider MARP.

a) MARP

From (3.1) and (3.2) we see that we have to find

$EV_\alpha^{T,k}$ = expected discounted cost in one cycle of length $\tau^{T,k} = \min(R^k,T)$ for $\alpha \ge 0$.

Now since the proof with time dependent costs is almost identical with the proof of Theorem 3.2, we just state the result:

$$EV_\alpha^{T,k} = E\int_0^T R_k(Y(x))e^{-\alpha x}dx + E\sum_{i=0}^{k} c_i(\tau^{T,k})I(Y(\tau^{T,k}) = i)e^{-\alpha\tau^{T,k}}$$

$$- E\sum_{i=k}^{M-1}\gamma_i(T)I(R^i > T)e^{-\alpha T}$$

$$= -\sum_{i=k}^{M-1}\int_0^T \bar{\bar{F}}^i(x)\beta_i^k(x)e^{-\alpha x}dx + \sum_{i=0}^{k}\int_0^T c_i(s)e^{-\alpha s}\,dQ_i^k(s) - \sum_{i=k}^{M-1}\gamma_i^k(T)e^{-\alpha T}\bar{\bar{F}}^i(T)$$

Thus we have found $B_\alpha^{T,k}$ and $B^{T,k}$ with time dependent costs and the minimization can now be done in a similar way as in the special case with $b_i(x) \equiv b_i$ and $c_i(x) \equiv c_i$ which we have discussed in detail in chapter 3.

b) <u>MBRP</u>

From Lemma 5.1 we see that if we find $w_\alpha^{T,k}$ = expected discounted cost in a block interval of length T, then $\bar{B}^{T,k}$ and $\bar{B}_\alpha^{T,k}$ are given.

If we proceed as in Theorem 5.2, we can show that

$$Ew_\alpha^{T,k} = Eb_\alpha^{T,k} + EC_\alpha^{T,k} + ED_\alpha^{T,k}$$

$$= -\sum_{i=k}^{M-1}\int_0^T \beta_i^k(x)e^{-\alpha x}D_i^k(x)dx$$

$$+ \sum_{i=0}^{k}\int_0^T e^{-\alpha u}c_i(u)[1+\int_0^{T-u} e^{-\alpha x}dM^k(x)]dQ_i^k(u)$$

$$- \sum_{i=k}^{M-1}\gamma_i^k(T)D_i^k(T)e^{-\alpha T}$$

where $D_i^k(x)$ is given in Theorem 5.2.

Note that we also have

$$Eb_\alpha^{T,k} = -\sum_{i=k}^{M-1}\int_0^T e^{-\alpha x}\beta_i^k(x)\bar{\bar{F}}^i(x)[1+\int_0^{T-x} e^{-\alpha u}dM^k(u)]dx$$

and

$$EC_\alpha^{T,k} = \sum_{i=0}^{k}\int_0^T e^{-\alpha u}c_i(u)dM_i^k(u) \quad \text{if } c_i(\cdot) \text{ has a derivative,}$$

see Appendix B:2 and b) of the proof of Theorem 5.2.

Minimizing $\bar{B}^{T,k}$ and $\bar{B}_{\alpha}^{T,k}$ is now similar to the special case with $b_i(x) \equiv b_i$ and $c_i(x) \equiv c_i$.

9. Multistate age replacement policy with non-negligable replacement times

<u>a</u> The replacement policy discussed here is identical with MARP except for the assumption of non-negligable replacement (repair) times.

Let $Z^{T,k}$ be a random variable representing the time needed for a repair of an arbitrary system when we replace at $\tau^{T,k} = \min(T,R^k)$.

If we start to repair when the system is in state i , then $Z^{T,k} = Z_i$. We will assume Z_i $i = 0,1,\dots,M$ is independent of R^j and θ_l $j = 0,1,\dots,M$, $l = 0,1,\dots,k$ and $EZ_i = z_i < \infty$. Furthermore, $z_0 \geqq z_1 \geqq \dots \geqq z_M > 0$.

Obviously we get a renewal process with length of one cycle, $\tau^{T,k} + Z^{T,k}$. Let $V^{T,k}$ be the cost of one cycle of length $\tau^{T,k} + Z^{T,k}$.

Then if $N^{T,k}$ is the expected long run average cost per unit time, it follows by Appendix A.2 that

$$N^{T,k} = \frac{EV^{T,k}}{E\tau^{T,k} + EZ^{T,k}}$$

From Theorem 3.2 we find $EV^{T,k}$ and $E\tau^{T,k}$.

(We assume no "occupancy" cost b_i under the repair time.)

Let $\quad v_k^k = -z_{k+1}$

$$v_i^k = z_i - z_{i+1} \qquad i = k+1,\dots,M-1$$

i.e. $\quad z_i = -\sum_{j=k}^{i-1} v_j^k .$

We now interpret $Z^{T,k}$ as the cost due to the replacement of the system. Then as in Theorem 3.2 we see that

$$EZ^{T,k} = \sum_{i=k}^{M-1} v_i^k F^i(T) + z_M + \sum_{i=0}^{k} z_i Q_i^k(T) .$$

We can now proceed as in the case of negligable replacement times, minimizing $N^{T,k}$.

Define $n^k(x)$ by

$$n^k(x) = \frac{\frac{d}{dx}EV^{x,k}}{\bar{\bar{F}}^k(x)+\sum_{i=k}^{M-1} v_i^k f^i(x)+\sum_{i=0}^{k} z_i q_i^k(x)}$$

if the densities exist.

Then we get a similar result as Theorem 3.7 with

$N^{T,k}$, $N^{\infty,k} = \dfrac{\sum_{i=k}^{M-1}\beta_i^k r_i + \sum_{i=0}^{k} c_i P_i^k}{r_k + \sum_{i=0}^{k} z_i P_i^k}$ and $n^k(x)$ in stead of $B^{T,k}$,

$B^{\infty,k}$ and $a^k(x)$.

b Barlow and Hunter (1960) consider an age replacement policy in the binary case when the object function is the fractional amount of functioning time over long intervals.

Let $\mathrm{Eff}^T(x)$ = expected fractional amount of time the system functions during $[0,x]$ and let

$$\mathrm{Eff}^T = \lim_{x\to\infty} \mathrm{Eff}^T(x) .$$

Eff^T is called the limiting efficiency and is used to find the optimal T.
Barlow and Hunter show that maximizing Eff^T is equivalent to minimizing the expected cost per unit time (as is done in Chapter 3) where now z_i $i = 0,1$ defined in a) is considered as the expected cost of performing replacements from state i.

Now we consider the MARP with control limit k. Since we may have several states in $G_k^c = \{k+1,\dots,M\}$ with different weights, we cannot generalize the definitions of Barlow and Hunter (1960) directly.

Let us define $Eff^{T,k}(x)$ as the expectation of the fraction represented by the "good" time (the time in G_k^c) in $[0,x]$ devided by a weighted sum of the time the system is in the different states (including the repair time).
More precisely

$$Eff^{T,k}(x) = E\left\{\frac{D^{T,k}(x)}{R^{T,k}(x)+Z^{T,k}(x)}\right\}$$

where $Z^{T,k}(x)$ = the time in $[0,x]$ the system is under repair,

$D^{T,k}(x)$ = the time in $[0,x]$ the system is in the states G_k^c

and $R^{T,k}(x) = \sum_{i=k+1}^{M} \{b_i \cdot \text{time in state } i \text{ in } [0,x]\}$.

Let $Eff^{T,k} = \lim_{x\to\infty} Eff^{T,k}(x)$. Then we want to find the T (and k) which maximizes $Eff^{T,k}$.
As in Barlow and Hunter (1960) page 94 we can show that

$$Eff^{T,k} = \frac{ED^{T,k}(\tau^{T,k})}{ER^{T,k}(\tau^{T,k})+EZ^{T,k}(\tau^{T,k})}$$

Now obviously

$$ED^{T,k}(\tau^{T,k}) = E\tau^{T,k} \quad \text{and}$$

$$ER^{T,k}(\tau^{T,k}) + EZ^{T,k}(\tau^{T,k}) = EV^{T,k} ,$$

where $EV^{T,k}$ is as in the numerator of (3.3) with $c_i(\gamma_i^k)$ replaced by $z_i(v_i^k)$.

Hence $$Eff^{T,k} = \frac{E\tau^{T,k}}{EV^{T,k}} = \frac{1}{\frac{EV^{T,k}}{E\tau^{T,k}}} = \frac{1}{B^{T,k}}$$

We see that maximizing $Eff^{T,k}$ is equivalent to minimizing $B^{T,k}$ where $B^{T,k}$ is as in (3.3) with c_i replaced by z_i . Thus the MARP which we have discussed in detail include non-negligable replacement times if we interpret c_i as the expected repair time when the repair starts from state i .

10. Examples of MARP and MBRP when $M = 2$ and $S = \{0,1\}$

Assume

$Q_0^0(t) = 1 - e^{-t}(1+t)$; i.e. a gamma distribution with parameters 1 and 2 .

$Q_1^1(t) = \frac{2}{3}(1-e^{-10t})$, $Q_0^1(t) = \frac{1}{3}(1-e^{-10t})$.

Then we have

$\bar{F}^0(t) = e^{-t}(1+t)$ $\qquad$ $\bar{F}^1(t) = \frac{2}{3}e^{-10t} + \frac{1}{3}e^{-10t} = e^{-10t}$

$f^0(t) = te^{-t}$ $\qquad$ $f^1(t) = 10e^{-10t}$

$M^0(t) = \frac{1}{4}[2t - 1+e^{-2t}]$ $\qquad$ $M^1(t) = 10t$.

See Barlow and Proschan (1965) page 57 for this result.

$$m^0(t) = \tfrac{1}{2}(1-e^{-2t}) \qquad m^1(t) = 10$$

$$r_0 = 2 \qquad r_1 = \frac{1}{10}$$

$$l_0 = 6 \qquad l_1 = \frac{1}{50}$$

The costs are $b_1 = 10$, $b_2 = 5$, $c_1 = c_2 = 1$ and $c_0 = 3$.

Thus

$$\beta_0^0 = -10 \qquad \beta_1^0 = 5$$

$$\beta_1^1 = -5$$

$$\gamma_0^0 = -1 \qquad \gamma_1^0 = 0$$

$$\gamma_1^1 = -1$$

We will assume the costs are the same for both MARP and MBRP.

10.1 MARP

Let $k = 0$.

Then
$$B^{T,0} = -\beta_0^0 + \frac{1}{\int_0^T \bar{F}(x)dx}\left[-\beta_1^0\int_0^T \bar{F}^1(x)dx + \gamma_0^0 F^0(T) + \gamma_1^0 F^1(T) + c_2 + c_0 Q_0^0(T)\right]$$

$$= -\beta_0^0 + \frac{1}{2-(T+2)e^{-T}}\left[-\beta_1^0\ \frac{1}{10}(1-e^{-10T}) + (c_0+\gamma_0^0)(1-e^{-10T}(1+T)) + c_2\right]$$

$$= 10 + \frac{1}{2-(T+2)e^{-T}}\left[-\frac{3}{2}e^{-10T} + \frac{5}{2} - 2Te^{-10T}\right]$$

$$B^{\infty,0} = -\beta_0^0 + \tfrac{1}{2}[-\beta_1^0\ \frac{1}{10} + c_0] = 11.25$$

$$a^0(x) = -\beta_0^0 - \beta_1^0\,\frac{e^{-9x}}{1+x} + (c_0-c_1)\frac{x}{1+x} = 10 + 2\frac{x}{1+x} - 5\frac{e^{-9x}}{1+x}$$

$$a^0(\infty) = -\beta_0^0 + (c_0-c_1) = 12$$

From Theorem 3.7 we see that there exists a finite minimum of $B^{T,0}$ if $a^0(\infty) > B^{\infty,0}$.

For our example this trivially holds.

Furthermore $a^0(x)$ is strictly increasing, so there exists a finite, unique minimum.

Plotting $B^{T,0}$ we find the optimal $T^* = 1.30$, $B^{T^*,0} = 11.13$.

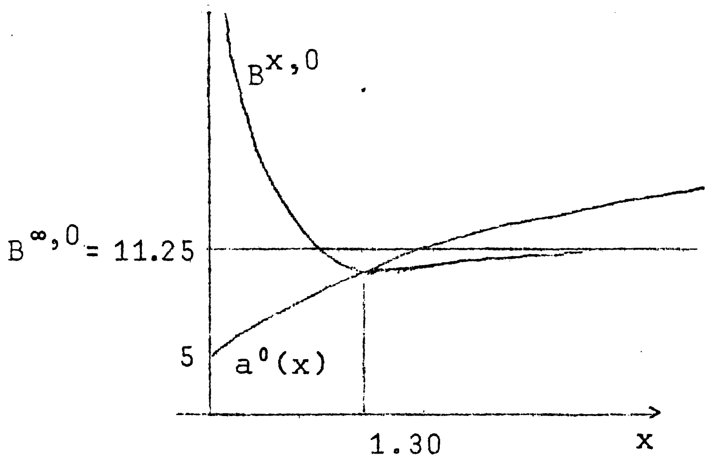

Fig. 10.1

Let $k = 1$. Then

$$B^{T,1} = b_2 + \frac{1}{\frac{1}{10}(1-e^{-10T})}[-c_2(1-e^{-10T}) + c_2 + (c_0\tfrac{1}{3}+c_1\tfrac{2}{3})(1-e^{-10T})]$$

$$= b_2 + 10(c_0\tfrac{1}{3}+c_1\tfrac{2}{3}) - 10c_2 + \frac{10c_2}{1-e^{-10T}} .$$

Clearly $T = \infty$ minimizes $B^{T,1}$.

In our example we find $B^{\infty,1} = 21.67$ which of course is greater than $B^{T^*,0}$. Hence the optimal MARP is

"Replace at min(1.30, R^0)".

10.2 MBRP

Let $k = 0$.

Since $d = (-\beta_0^0 r_0 - \beta_1^0 r_1)\frac{l_0}{2r_0^2} + \frac{1}{2}\frac{1}{r_0}(\beta_0^0 l_0 + \beta_1^0 l_1) - (\gamma_0^0 r_0 + \gamma_1^0 r_1)\frac{1}{r_0} + c_0\frac{l_0}{2r_0^2} - \frac{c_0 r_0}{r_0}$

$$= (10\cdot 2 - 5\tfrac{1}{10})\tfrac{6}{8} + \tfrac{1}{2}\,\tfrac{1}{2}(-10\cdot 6 + 5\tfrac{1}{50}) - (-12 + 0\tfrac{1}{10})\tfrac{1}{2} + 3\tfrac{6}{8} - 3 = -0.10 < 0 ,$$

there exists a finite minimum by Theorem 5.10 a) .

$$\bar{a}^0(x) = -\beta_1^0 - \beta_1^0 D_1^0(x) + c_0 m^0(x) = 10 - 5D_1^0(x) + 3[\tfrac{1}{2}(1-e^{-2x})] ,$$

where $D_1^0(x) = \bar{F}^1(x) + \int_0^x \bar{F}^1(x-u)dM^0(u)$

$$= \frac{1}{20} + \frac{81}{80} e^{-10x} - \frac{1}{16} e^{-2x}$$

We see that $\bar{a}^0(x)$ is strictly increasing. Thus by Theorem 5.10 there exists a unique minimum.

Plotting $\bar{B}^{x,0} = b_1 - \frac{1}{x}(\beta_1^0 A_1^0(x) + c_0 M_0^0(x) - \gamma_0^0)$,

where $A_1^0(x) = \int_0^x D_1^0(u)du$ and $M_0^0(x) = M^0(x) = \frac{1}{4}(2x-1+e^{-2x})$,

we find the optimal x : $x^* = 1.30$ and $\bar{B}^{x^*,0} = 11.16$.

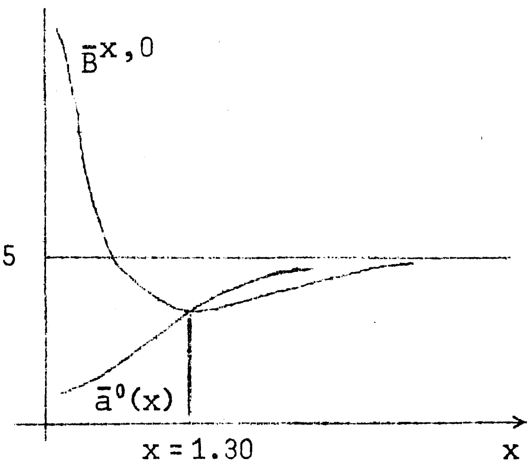

Fig. 10.2

Let $k = 0$.

$$\bar{B}^{x,1} = b_2 + \frac{1}{x}[c_0 M_0^1(x) + c_1 M_1^1(x) - \gamma_1^1]$$

where $M_0^1(x) = \frac{1}{3}[\int_0^x (1+M^1(x-u)dF_0^1(u)]$

$$= \frac{1}{3}[\int_0^x (1+M^1(x-u))dF^1(u)]$$

$$= \frac{1}{3}[F^1(x) + \int_0^x M^1(x-u)dF^1(u)]$$

$= \frac{1}{3}M^1(x) = \frac{10}{3}x$ since $M^1(\cdot)$ satisfies the renewal equation.

$M_1^1(x) = \frac{2}{3}10x$ since $M_0^1(x) + M_1^1(x) = M^1(x)$.

Hence $\bar{B}^{x,1} = 5 + 10 + \frac{2}{3}10 + \frac{1}{x} = 21.67 + \frac{1}{x}$ and the optimal rule, when the control limit is 1 , is $x = \infty$ and $B^{\infty,1} = 21.67$.

We have found the optimal MBRP :

"Replace at failure (0) and at $n \cdot 1.30$, $n = 1,2,\ldots$ "

Since $B^{1.30,0} = 11.13 < 11.16 = \bar{B}^{1.30,0}$, the optimal MARP is better than the optimal MBRP in this example.

Acknowledgement

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Appendix A

A.1 Wald's Equation

If $X_1, X_2, \ldots$ are independent and identically distributed random variables having finite expectations and N is an integer valued positive random variable such that the event "$N = n$" is independent of $X_{n+1}, X_{n+2}, \ldots$ for all $n = 1,2,\ldots,$ then

$$E \sum_{i=1}^{N} X_i = EN\, EX .$$

Proof

See Barlow and Proschan (1975) page 169-170.

Corollary

Let $\{X_i\}_{i=1}^{\infty}$, $\{Y_i\}_{i=1}^{\infty}$ be renewal processes with $E|Y_1| < \infty$ such that the pair $(X_1,Y_1),(X_2,Y_2),\ldots$ are mutually independent.
Let $N(t) = \sup\{n: \sum_{i=1}^{n} X_i \leq t\}$ and $M(t) = EN(t)$.
Then $E \sum_{n=1}^{N(t)+1} Y_n = EY_1(M(t)+1)$.

Proof

Let $S_n = \sum_{i=1}^{n} X_i$. Then

$N(t)+1 = n \iff N(t) = n-1 \iff S_{n-1} \leq t$ and $S_n > t$.

It follows that the event "$N(t)+1 = n$" is independent of $X_{n+1}, X_{n+2}, \ldots$. The result is now a consequence of Wald's equation.

A.2

Assume we are in same situation as in the corollary to Wald's equation.

If $Y(t) = \sum_{n=1}^{N(t)} Y_n$ and $EX_1 < \infty$, then $\dfrac{EY(t)}{t} \xrightarrow[t\to\infty]{} \dfrac{EY_1}{EX_1}$.

Proof

Ross (1970) page 52-53.

Remark

Y_n may be regarded as a reward or cost at each renewal generated by the renewal process $\{X_i\}_{i=1}^{\infty}$. If we assume continuously reward (cost), then the total reward (cost) up to time t is

$Y(t) = \sum_{n=1}^{N(t)} Y_n + \delta_{N(t)}$ where

Y_n is the reward (cost) associated with the n-th renewal cycle and

$\delta_{N(t)}$ is the reward (cost) in $(S_{N(t)}, t]$.

We will show that $\frac{EY(t)}{t} \xrightarrow[t\to\infty]{} \frac{EY_1}{EX_1}$ also in this case. See Ross (1970) page 53-54.

Assume first the reward (cost) is positive. Then

$$\frac{1}{t} E \sum_{n=1}^{N(t)} Y_n \leq \frac{EY(t)}{t} \leq \frac{1}{t} E \sum_{n=1}^{N(t)+1} Y_n$$

and the result follows by A.2, the corollary to Wald's equation and A.4.

The case when rewards (costs) are negative is treated similarly.

Then we have

$$E \sum_{n=1}^{N(t)+1} Y_n \leq \frac{EY(t)}{t} \leq \frac{1}{y} E \sum_{n=1}^{N(t)} Y_n .$$

Assume now general rewards (costs). Let

$Y_n = Y_n^+ - Y_n^-$, where

Y_n^+ is the total positive reward (cost) associated with the n-th renewal cycle and

$-Y_n^-$ is the total negative reward (cost) associated with the the n-th renewal cycle.

A similar partition is made for $\delta_{N(t)}$.

It follows that

$$Y(t) = \sum_{n=1}^{N(t)} Y_n^+ + \delta^+_{N(t)} - \left(\sum_{n=1}^{N(t)} Y_n^- + \delta^-_{N(t)}\right) \quad \text{and}$$

$$\frac{EY(t)}{t} = \frac{1}{t} E\left(\sum_{n=1}^{N(t)} Y_n^+ + \delta^+_{N(t)}\right) - \frac{1}{t} E\left(\sum_{n=1}^{N(t)} Y_n^- + \delta^-_{N(t)}\right)$$

$$\xrightarrow[t\to\infty]{} \frac{EY_1^+}{EX_1} - \frac{EY_1^-}{EX_1} = \frac{EY_1}{EX_1}$$

Remark 2.

Under MARP $X_i = \tau_i^{T,k} = \min(T, R_i^k)$ and Y_i is the cost associated with the "i-th system".

A.3

If $g(t) = h(t) + \int_0^t g(t-x)dF(x) \quad t \geq 0$ and h is bounded,

then $\quad g(t) = h(t) + \int_0^t h(t-x)dM(x)$

where $M(x) = \sum_{n=1}^{\infty} F_{(n)}(x)$ and $F_{(n)}$ is the n-fold convolution of F.

Proof

Ross (1970) page 35.

A.4 The Elementary Renewal Theorem. (Ross (1970) page 40)

$$\frac{M(t)}{t} \to \frac{1}{\int_0^\infty x dF(x)} \quad \text{as} \quad t \to \infty \quad \text{where} \quad M(t) = \sum_{n=1}^{\infty} F_{(n)}(t) .$$

A.5 The Key Renewal Theorem. (Feller (1971))

If F is not lattice,* with mean μ and if $h(t)$ is directly Riemann integrable (Feller (1971) page 362) then

$$\lim_{t\to\infty} \int_0^t h(t-x)dM(x) = \frac{1}{\mu} \int_0^\infty h(t)dt$$

* X non-negative random variable is said to be lattice if there exists $d \geq 0$ such that $\sum_{n=0}^{\infty} P\{X=nd\} = 1$.

Remark

Sufficient conditions for h to be directly Riemann integrable, Feller (1971) chapter XI.

a) i) $h(t) \geq 0$

ii) $h(t)$ is decreasing

iii) $\int_0^\infty h(t)dt < \infty$

b) $h(t) \geq 0$, h bounded and continuous,

$\sum_n u_n < \infty$ where $u_n = \max_{n-1 \leq t < n} h(t)$

c) $\int_0^\infty |h'(t)|dt < \infty$

A.6

The renewal function $M(t) = \sum_{n=1}^\infty F_{(n)}(t)$ is finite for all t and converges uniformly on finite intervals. It follows that $M(t)$ is right-continuous and continuous if $F(t)$ is continuous.

Manifestly, $M(t)$ is an increasing function of t. Furthermore $\lim_{t\to\infty} M(t) = \infty$.

We have listed some properties of $M(t)$ stated in Karlin (1975) page 182.

From Feller (1971) page 367 we get the renewal density theorem: If F has a bounded density f, $\int_0^\infty xf(x)dx < \infty$, $\lim_{x\to\infty} f(x) = 0$ (or F has a directly Riemann integrable density f), then $M(\cdot)$ has a density $m(\cdot)$ and

$$m(x) \to \frac{1}{\int_0^\infty tf(t)dt} \quad \text{as} \quad x \to \infty .$$

Smith (1954) page 42, shows the same theorem under the assumption: $f(x) \to 0$ as $x \to \infty$ and $\int_0^\infty f(x)^p dx < \infty$ for some $p > 1$.

Appendix B

B.1

Let $\{\tau_n\}_{n=1}^{\infty}$ be a renewal process (with $\tau_0 = 0$), distributed as τ for $n \geq 1$,

$V_n(\tau_n) = V_n^+(\tau_n) - V_n^-(\tau_n)$ the discounted cost in the n-th cycle where $V_n^+(\cdot)$ $(-V_n^-(\cdot))$ is the total positive (negative) cost in the n-th cycle.

Then the total expected, discounted cost is

$$E\left[\sum_{n=1}^{\infty} e^{-\alpha(\tau_1+\cdots+\tau_{n-1})} V_n^+(\tau_n) - \sum_{n=1}^{\infty} e^{-\alpha(\tau_1+\cdots+\tau_{n-1})} V_n^-(\tau_n)\right]$$

$$= \sum_{n=1}^{\infty} E\, e^{-\alpha(\tau_1+\cdots+\tau_{n-1})} V_n^+(\tau_n) - \sum_{n=1}^{\infty} E\, e^{-\alpha(\tau_1+\cdots+\tau_{n-1})} V_n^-(\tau_n)$$

$$= \sum_{n=1}^{\infty} (Ee^{-\alpha\tau})^{n-1} EV_n^+(\tau_n) - \sum_{n=1}^{\infty} (Ee^{-\alpha\tau})^{n-1} EV_n^-(\tau_n)$$

$$= \frac{EV^+(\tau) - EV^-(\tau)}{1-Ee^{-\alpha\tau}} = \frac{EV(\tau)}{1-Ee^{-\alpha\tau}}$$

We have used monotone convergence theorem and the fact that the cost in the n-th cycle is independent of τ_i, $i < n$.

Note that under MARP (MBRP) $\tau = \min(T, R^k)$ $(\tau \equiv T)$.

There is an alternative way to find the total expected, discounted cost.

Since the process restarts at the time for the first renewal, the total expected, discounted cost, B_α must satisfy the equation

$$B_\alpha = EV_1(\tau_1) + Ee^{-\alpha\tau_1} B_\alpha$$

(this is easily seen by conditioning on the time for the first renewal)

Hence $$B_\alpha = \frac{EV(\tau)}{1-Ee^{-\alpha\tau}}.$$

B.2

Assume $v(\cdot)$ is either an increasing, right-continuous real valued function or a stochastic process which is right-continuous and increasing with probability one. Assume $v(0^-) = 0$.
Let $c(\cdot)$ be a real valued function, and assume we may write

$$c(x) = \int_0^x c'(u)du + c(0) .$$

Then we state the following formula of integration by parts.
See Gikman and Shorohod (1979) page 18.

$$\text{(b.2)} \qquad \int_0^T c(x)dv(x) = v(T)c(T) - \int_0^T c'(t)v(t)dt .$$

Proof

$$\begin{aligned}
\int_0^T c(x)dv(x) &= c(0)\cdot v(T) + \int_0^T \int_0^T c'(t)I(t,x : t\leq x)dt\ dv(x) \\
&= c(0)\cdot v(T) + \int_0^T \int_0^T c'(t)I(t,x : t\leq x)dv(x)dt \\
&= c(0)\cdot v(T) + \int_0^T c'(t) \int_0^T dv(x)dt \\
&= c(0)\cdot v(T) + v(T) \int_0^T c'(t)dt - \int_0^T c'(t)v(t)dt \\
&= c(0)\cdot v(T) + v(T)(c(T)-c(0)) - \int_0^T c'(t)v(t)dt \\
&= v(T)c(T) - \int_0^T c'(t)v(t)dy \qquad \text{Q.E.D.}
\end{aligned}$$

Remark

1 $\int_0^T c(x)dv(x)$ is a Lebesgue-Stieltjes integral and when $v(\cdot)$ is a stochastic process which is right-continuous and increasing with probability one, then this integral exists with probability one.

Let $v(\cdot)$ be defined on the probability space $(\Omega,\mathcal{A},P)$. Then $v(x,w)$ generates for a fixed $w \in \Omega$ a measure $v(A)$ on $[0,\infty)$ such that $v(a,b] = v(b) - v(a)$.

2 Assume $v(\cdot)$ is a right-continuous, increasing stochastic process with $Ev(x) < \infty \quad \forall x$. Then

$$E\int_0^T c(x)dv(x) = \int_0^T c(x)d(Ev)(x)$$

To see this we use (b.2) twice and Fubini.

$$\begin{aligned} E\int_0^T c(x)dv(x) &= E[v(T)c(T) - \int_0^T c'(t)v(t)dt] \\ &= Ev(T)c(T) - \int_0^T c'(t)Ev(t)dt \\ &= \int_0^T c(x)d(Ev)(x) . \end{aligned}$$

Note that $Ev(x)$ is an increasing, right-continuous function.

B.3

Let $\varphi(\cdot)$ be a binary coherent structure function and let $x = (x_1,x_2,\ldots,x_n)$ be given such that $\varphi(x) = 1$. Assume $x < y$ ($y_i \geq x_i \quad \forall i$ with $y_i > x_i$ for some i).

Let $v(x) = \sum_{i=1}^{n} \lambda_i x_i (1 - \varphi(0_i,x))$ *

Then $v(x) \geq v(y)$.

Proof

$$\begin{aligned} v(x) - v(y) &= \sum_{i=1}^{n} \lambda_i x_i (1-\varphi(0_i,x)) - \sum_{i=1}^{n} \lambda_i y_i (1-\varphi(0_i,y)) \\ &= \sum_{\substack{i=1 \\ x_i = 1, y_i = 1}}^{n} \lambda_i (\varphi(0_i,y) - \varphi(0_i,x)) + \sum_{\substack{i=1 \\ x_i = 0, y_i = 1}}^{n} \lambda_i (\varphi(0_i,y) - 1) \end{aligned}$$

* Notation $(\cdot_i,x) = (x_1,\ldots,x_{i-1},\cdot,x_{i+1},\ldots,x_n)$

Now since $1 \geq \varphi(0_i, y) = \varphi(x_i, y) \geq \varphi(x) = 1$ if $x_i = 0$, we see that the last sum equals 0, and obviously $\varphi(0_i, y) - \varphi(0_i, x) \geq 0$. Thus $v(x) - v(y) \geq 0$.

The proof is due to Bent Natvig.

Remark

This obviously gives that $v_j(X(t))$ defined on page 38 is an increasing function of time t.

References

Abdel-Hameed, M. and Shimi, IN. (1978) Optimal replacement of damaged devices. J. Appl. Prob., 15, 153-161.

Barlow, R.E. and Hunter, L. (1960 c). Optimum preventive maintenance policies. Operations Res., V.8, No 1, 99-100.

Barlow, R.E. and Proschan, F. (1965) Mathematical Theory of Reliability. Wiley, New York.

Barlow, R.E. and Proschan, F. (1975). Statistical Theory of Reliability and Life Testing. Probability Models Holt, Rinehart and Winston, New York.

Beichelt, F. and Fischer, K. (1980) General failure model applied to preventive maintenance policies. IEEE Trans. Reliability, R-29, No 1, April, 39-41.

Berg, M. (1980). A marginal cost analysis for preventive replacement policies. European Journal of operational Research, 4, 136-142.

Berg, M. and Epstein, B. (1976). A modified block replacement policy. Naval Res. Log. Quart. 23, 15-24.

Berg, M. and Epstein, B. (1978) Comparison of age, block and failure replacement policies. IEEE Trans. Rel. R-29, No 1, April, 25-28.

Bergman, B. (1978). Optimal Replacement under a general failure model. Adv. Appl. Prob. 10, 431-451.

Bergman, B. (1980 a) A note on the completeness of stationary replacement rules. J. Appl. Prob. 17, no 1, 178-186.

Bergman, B. (1980). On some recent advances in replacement theory. From the symposium "Tillförlighetsdager på Chalmars 5-6 mai 1980", Chalmars Institute of Technology, Gothenburg.

Bobos, A.G. and Protonotarios, E.N. (1978) Optimal systems for equipment maintenance and replacement under Markovian deterio-ation. European Journal of Operational Research 2, 257-264.

Cleroux, R. and Hanscom, M. (1974) Age replacement with adjustment and depreciation cost and interest charges. Technometrics, Vol 6, No 2, 235-239.

Cox, E.R. (1963) Renewal Theory, John Wiley and Sons, New York.

Derman, C. (1963 c) On optimal replacement rules when changes of state are Markovian, Chap. 9, Mathematical Optimazation, Techniques, ed. Richard Bellman, Univ. of Calif. Press, Berkely and Los Angeles, pp. 201-210.

Derman, C. and Sacks, J. (1978) Replacement of periodically inspected equipment. Naval Res. Log. Quart. V. 7, No 4, 597-607.

Feldman, R.M. (1976) Optimal replacement with Semi-Markov shock models. J. Appl. Prob. 13, 108-117.

Feller, W. (1968) An Introduction to Probability Theory and its Applications, Vol. I. 3rd ed., Wiley, New York.

Feller, W. (1971) An Introduction to Probability Theory and its Applications, Vol II, second ed., Wiley, New York.

Flehinger, B.J. (1962) A general model for the reliability of systems under various preventive maintenance policies. Annals of Math. Statistics. Vol. 33, No. 1, 137-156.

Fox, B. (1966) Age replacement with discounting. Operations Res. Vol. 14, 533-537.

Gertsbach, I. (1977) Models for Preventive Maintenance. North Holland Publ. Comp., Amsterdam.

Gihman, I.I. and Shorohod, A.V. (1979) The Theory of Stochastic Processes III. Springer.Verlag, New York Inc.

Jain, A. and Nair, K.P.K. (1974) Comparison of replacement strategies for items that fail. IEEE Trans. Rel. Vol. R 23, No 4, Oct. 247-251.

Jorgenson, D., McCall, J. and Radner, R. (1967) Optimal Replacement Policies Rand McNally.

Kaio, N. and Osaki, S. (1979) Optimum age replacement policy with two failure models. R.A.I.R.O. Operat. Res. 1205-1208.

Kander, Z. (1978) Inspection policies for deteriorating equipment characterized by N quality levels. Naval Res. Log. Quart,. 25, 243-255.

Kao, E.P.C. (1973) Optimal replacement rules when changes of states are Semi-Markovian. Opns. Res. 21, 1231-1249.

Karlin, S. and Taylor, H.M. (1975) A First Course in Stochastic Processes, sec. ed., Academic Press, New York.

Kolesar, P. (1966) Minimum cost replacement under Markovian deterioration. Management Science 12, 694-706.

Laha, R.G. and Rohatgi, V.K. (1979) Probability Theory. John Wiley & Sons, New York.

McCall, J.J. (1965) Maintenance policies for stochastically failing equipment: A Survey, Management Science, Vol. 11, No. 5, 493-524.

Mine, H. and Nakagawa, T. (1978) Age replacement model with mixed failure times. IEEE Trans. Rel. Vol. 27, No 2, 173.

Nakagawa, T. (1979) A summary of block replacement policies. R.A.I.R.O. Vol. 13, 4, 351-361.

Natvig, B, (1980) Two suggestions of how to define a multistate coherent system, Statistical Res. Report, Inst. of Math. Univ. of Oslo.

Nummelin, E. (1978) A general failure model: Optimal replacement with state dependent replacement and failure costs. Report - MTKK-MAT-A134. Helsinki Univ. of Technology, Inst. of Math., SF-02150, Espov. 15, Finland.

Osaki, S. and Nakagawa, T. (1975) A note on age replacement, IEEE Trans. Rel. Vol. R-24, No 1, 92-94.

Pierskalla, W.P. and Voelker, J.A. (1976) A survey of maintenance models: The control and surveillance of deteriorating system. Naval Res. Log. Quart. 23, 353-388.

Ran, A. and Rosenlund, S. (1976) Age replacement with discounting for a continuous maintenance cost model. Technometrics, Vol. 18, No 4, 459-465.

Ross, S. (1970) Applied Probability Models with Optimazation Applications, Holden-Day.

Ross, S. (1971) Infinitesimal look-ahead stopping rules. The Annals of Math. Statistics, Vol. 42, 297-303.

Scheaffer, R.L. (1971) Optimum age replacement policies with an increasing cost factor. Technometrics, Vol. 13, No. 1,139-149.

Sethi, D. (1977) Opportunistic replacement policies. The Theory and Applications of Reliability, Vol. 1, Edit. Tsokos, C.P. and Shimi, I.N., pp. 433-447, Academic Press inc.

Sivazlian, B.E. (1973) On a discounted replacement problem with arbitrary repair time distribution. Management Sci. Vol. 19, No. 11, 1301-1309.

Smith, W.L., (1954) Asymptotic renewal theorems, Proc. Roy. Soc. Edinburgh, Sect. A, V. 64, Pt. I (No.2), pp. 9-48.

Tango, T. (1978) Extended block replacement policy with used items. J. Appl. Prob. 15, 560-572.

Taylor, H.M. (1975) Optimal replacement under additive damage and other failure models. Naval Res. Log. Quart. 22, 1-18.

Tilquin, C. and Cléroux, R. (1975) Periodic replacement with minimal repair at failure and general cost function. J. Statis. Comput. Simul.,Vol. 4, 63-77.

Tilquin, C. and Cléroux, R. (1975) Periodic replacement with minimal repair at failure and adjustment cost. Naval Res. Log. Quart. 22, 243-254.

Tilquin, C. and Cléroux, R. (1975) Block replacement policies with general cost structures. Technometrics, Vol. 17, No 3, 291-298.

Wolff, M.R. and Subramanian, R. (1974) Optimal readjustment intervals, Operat. Res. 22, 191-197.

Zuckerman, D. (1978) Optimal replacement policy for the case where the damage process is a one sided Lévy process. Stochastic Processes and their Applications, 7, 141-151.

Zuckerman, D. (1978) Optimal stopping in a Semi-Markov shock model. J. Appl. Prob. 15, 629-634.